Public On-Demand Transportation: A Markovian Continuous Approximation Model

Daniel F. Silva
Department of Industrial and Systems Engineering, Auburn University, 3301 Shelby Center, AL 36849, silva@auburn.edu

Alexander Vinel
Department of Industrial and Systems Engineering, Auburn University, 3301 Shelby Center, AL 36849, alexander.vinel@auburn.edu

Bekircan Kirkici
Department of Industrial and Systems Engineering, Auburn University, 3301 Shelby Center, AL 36849, bzk0054@auburn.edu

1. Motivation and Contribution

While only 11% of people in the United States use public transit on a daily basis, Americans who are lower-income, black, Hispanic or immigrants are especially likely to rely on this form of transportation, with up to 25% daily users for those groups (Anderson 2016). However, especially in the U.S., public transit agencies are facing increasing pressures from the growth of ride-sharing platforms, such as Uber and Lyft. Graehler Jr, Mucci, and Erhardt (2019) suggest that such services entering a city can cause a 1.3% decrease in rail and a 1.7% decrease in bus ridership. The reduction in ridership can then lead to reduced public funding, which in turn leads to a decline in service quality. At the same time, new ride-sharing-platform technologies provide public transit systems with an opportunity to adapt and improve their services. Many cities are exploring options for incorporating app-based services into their transit offerings, with varied level of success (Bliss 2018). One avenue for such applications is often referred to as *on-demand transit*. This is a fixed price public service offered by the city that (i) is available to everyone in the service region, (ii) can be requested immediately before the service is needed, (iii) aims to meet demand with a maximum delay guarantee, and (iv) may or may not pool riders.

On-demand public transit has unique features, such as equity and fairness requirements, which are difficult to measure and that for-profit platforms do not currently address. There has been a lot of research focused on optimally dispatching and routing on-demand transportation operations (Agatz et al. 2012). There has also been research on using continuous approximation to measure performance of transportation systems in terms of average cost or route duration (Franceschetti,

Jabali, and Laporte 2017). However, there has been little work on obtaining distributional information of performance metrics under different operating parameters and configurations. This information is crucial to understanding equity and fairness, which are usually measured in terms of probabilities or quantiles.

In this paper we propose a first-of-its-kind model of a specific, single-vehicle, on-demand public transit system, which merges a Markov chain analysis with a continuous approximation approach to estimate complicated performance metrics such as distribution of passengers per trip and probability of being rejected from the system. We define several performance metrics analytically and also prove a few asymptotic properties with respect to the demand. Additionally, we use this model in a case study, based on a pilot on-demand transit project that was implemented in a major U.S. metropolitan area. Finally, the model developed can be viewed as a demonstration of how continuous approximation models can be constructed based on a distributional result.

2. Mathematical Model

2.1. Assumptions

Our goal is to model an idealized on-demand transportation system for first-mile/last-mile operation. For clarity of exposition, we assume only first-mile operations, in order to analytically derive some estimates for the system's performance. We consider the following assumptions on the operation of the on-demand system. There is a single vehicle with capacity K, based at a dedicated station (such as an end-of-line bus stop). The service area is a bounded 2-D region around the station. Customers arrive following a Poisson process with rate λ. Customer locations are random and follow some predetermined distribution on the region that is independent from arrival time. We assume, without loss of generality, that the speed of the vehicle is equal to 1. Pick-ups and drop-offs are instantaneous. The vehicle leaving the depot will pick-up up to K currently waiting customers, but will not change its route to accommodate any newly incoming customers. Instead, all (up to K) such customers will be served during the next vehicle trip. A new customer is rejected whenever there already are K customers waiting for the vehicle to complete the current trip.

2.2. Embedded Markov Chain

We model the system as a discrete-time Markov Chain $\{X_n,\, n=0,1,2,...\}$. We observe the system at specific epochs, n, namely at the beginning of each trip. We define X_n as the number of customers on the n-th trip. We assume that if during the n-th trip there are k customer requests, then $X_{n+1}=\min\{k,K\}$. If more than K requests come in during a trip, those requests are rejected by the system. On the other hand if zero requests come in during the n-th trip, then $X_{n+1}=0$. We call this state a "virtual trip" which ends immediately as soon as the next request arrives, so necessarily if $X_n=0$, then $X_{n+1}=1$. The state space for X_n is $\mathcal{S}=\{0,1,...,K\}$. If $X_n>0$, then

state transition is determined by the number of new customers, arriving during the n-th trip, which in turn is determined by two sources of uncertainty: the Poisson arrival process for the customers, and random trip length to pick-up the passengers (of the current trip). Let us define as Y^i the random variable representing the trip-length between i random passengers, i.e., the length of a TSP tour between i independent locations distributed on the service area and the depot. We will denote the corresponding pdf as $f^i(\cdot)$. Define the function $\mathcal{P}(\lambda,k)$ as the probability of a random variable with distribution $Poisson(\lambda)$ taking value k, if $k<K$, and the probability that a $Poisson(\lambda)$ random variable takes a value greater than or equal to K, when $k=K$. Then, state transitions can be calculated as $p_{01}=1$ and $p_{0k}=0$, $\forall k\neq 1$, and $p_{ik}=\int_{-\infty}^{\infty}\mathcal{P}(\lambda y,k)f^i(y)dy$, $\forall i>0,\forall k$. We denote $\pi_k(\lambda,K)$ as the stationary probabilities for this DTMC for given λ and K.

2.3. Trip Length Distribution

Note that calculating $\pi_k(\lambda,K)$ requires specific knowledge of the distribution of Y^i. An important observation, is that this distribution no longer depends on the system parameters, and is determined by the spacial distribution of customer requests only. Hence, it can be either determined analytically, or with a simple Monte-Carlo simulation. For example, consider a disk of radius r. If customers are distributed uniformly, then Y^i can be approximated as follows. Clearly, $Y^1\in\text{Triangular}(0,r,r)$. As shown in Vinel and Silva (2018) for $i\geq 2$, Y^i is well-described by the Normal distribution. Hence the transition probabilities can be written as $p_{1k}=\int_0^r\mathcal{P}(\lambda y,k)\frac{2y}{r^2}dy$ and $p_{ik}=\int_{-\infty}^{\infty}\mathcal{P}(\lambda y,k)f^i(y)dy$. In this case f^i is the pdf of a $N(\mu_i,\sigma_i^2)$ random variable, where according to Vinel and Silva (2018), the parameters are $\mu_i=\left(0.768\sqrt{i}\right)(r\sqrt{\pi})$ and $\sigma_i=\left(0.4985+\frac{0.1826}{\sqrt{i}}\right)(r\sqrt{\pi})$. However, this allows Y to take negative values, which can lead to erratic behavior of X_n. So we truncate and re-scale Y, obtaining $p_{ik}=\int_0^{\infty}\mathcal{P}(\lambda y,k)\frac{f^i(y)}{1-F^i(0)}dy$, where $f^i()$ and $F^i()$ are the pdf and cdf of a $N(\mu_i,\sigma_i^2)$.

Under fairly general trip-length distributions we can show that the result below holds. We can then use π_k to obtain a wide variety of performance metrics.

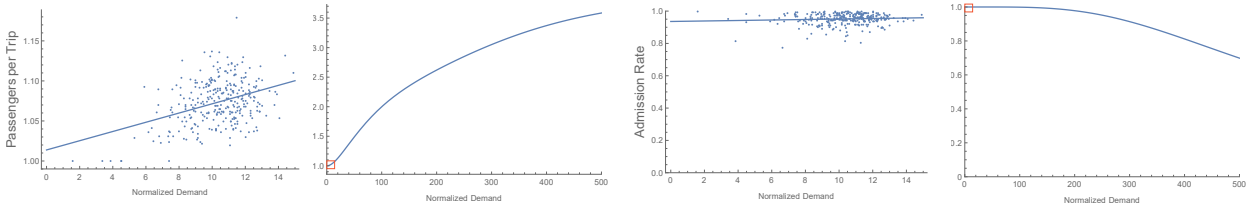
PROPOSITION 1. *$\pi_k(\lambda,K)$ exist and are unique for all $\lambda>0$, $K\geq 1$. $\pi_0(\lambda,K),\pi_1(\lambda,K)\to\frac{1}{2}$ as $\lambda\to 0$, and $\pi_K(\lambda,K)\to 1$ as $\lambda\to\infty$. $\pi_0(\lambda,K)$ and $\pi_K(\lambda,K)$ are monotone in λ.*

3. Case Study

We consider data from a pilot on-demand transit project in a major U.S. metropolitan area. We calculate performance metrics using our model with the parameters used by the real-world system and compare to the observed outcomes. For example, the average number of passengers per actual trip is given by $Num(\lambda,K)=\sum_{k=1}^{K}k\frac{\pi_k}{1-\pi_0}$ and the fraction of customers admitted (probability of being admitted) is given by:

$$Admit(\lambda,K)=\frac{\pi_0}{\lambda LT}+\sum_{i=1}^{K}\frac{\pi_i\mu_i}{LT}\left(\sum_{k=1}^{\infty}\frac{\min\{k,K\}}{k}\frac{\mathcal{P}(\lambda\mu_i,k)}{1-\mathcal{P}(\lambda\mu_i,0)}\right).$$

where LT represents the average trip length. In Figures 1a and 1b, we compare the average passengers per ride between the data and theoretical calculation as a function of demand. Similar graphs for the admission rate are given in Figures 1c and 1d. The red square in the Figures 1b and 1d represents the area of demand which corresponds to the observed data.



(a) Num (data) (b) Num (theoretical) (c) Admit (data) (d) Admit (theoretical)

Figure 1 Performance metrics

Observe that the theoretical graphs predict that the real system has been operating on an extremely low demand volume. Further, the admission rate curve is relatively flat, i.e., the system could accommodate significantly more customers without undermining this particular metric. Average number of passengers, on the other hand, is more sensitive to demand increase. Overall, a more careful analysis, including a larger variety of customer-facing measures is required in order to evaluate true system capacity and then prescribe optimal design.

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