A Branch-Price-and-Cut Algorithm for Robust Vehicle Routing under Demand Uncertainty

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1. Introduction

The Vehicle Routing Problem (VRP) is one of the most highly studied combinatorial optimization problems in the area of Supply Chain and Logistics. Recently, the VRP under demand uncertainty has attracted considerable attention. It is motivated by the fact that customer demands are often not readily available at the time the distributor needs to commit to some routing plans. *Robust Optimization* (RO) (Ben-Tal, El Ghaoui, and Nemirovski 2009) can be a promising approach to dealing with uncertainty in customer demands, guaranteeing the immunity of the optimal routing design from all demand realizations in a given uncertainty set.

In the realm of deterministic problems, the *Branch-Price-and-Cut* (BPC) approach has recently emerged as the state-of-the-art algorithm to solve deterministic VRPs (Toth and Vigo 2014). This approach starts with a subset of feasible routes and entails the solution of *pricing subproblems* to dynamically introduce feasible vehicle routes with the potential of reducing the objective value. Whereas lots of literature has been devoted on how to address deterministic VRPs, there have been only a few works combining RO with BPC to tackle VRPs under uncertainty. The first approach was originally introduced in Lee, Lee, and Park (2012), and was more recently revisited by Munari et al. (2019); here, the pricing subproblem was formulated as a robust *shortest path problem with resource constraints* (SPPRC), so as to generate routes that are robust feasible with respect to demand uncertainty. The authors of the above works demonstrated that the resulting robust SPPRC could be efficiently solved for the commonly used cardinality-constrained uncertainty set. Another approach was proposed in Pessoa et al. (2018) and Lu and Gzara (2019), where

the authors transformed a robust SPPRC into finitely many deterministic SPPRCs, avoiding the burden of solving robust SPPRCs. The former work showed that the transformation could be achieved for two popular uncertainty sets from the RO literature.

Though both approaches seem quite promising, they take advantage of the structure of uncertainty sets and hence can not be generalized to many commonly used uncertainty sets from the RO literature. Therefore, it is of great necessity to develop a novel approach that can work for solving VRPs under various, general types of uncertainty sets. The work of Gounaris, Wiesemann, and Floudas (2013) proposed a cutting-plane idea to enforce robust feasibility and embedded it into a branch-and-cut framework for addressing robust VRPs under demand uncertainty. This motivates us to develop a novel approach of combining the robust cutting-plane idea with the state-of-the-art BPC advances in deterministic settings, so as to address VRPs under a variety of uncertainty sets.

2. Problem Definition

The classical variant of the *Capacitated Vehicle Routing Problem* (CVRP) calls for determining minimum-cost routes to be traversed by a fleet of K homogeneous vehicles of capacity Q so that a set of customers V_c with known demands q is served. We now consider that the demand vector q is uncertain and can take any value from a given uncertainty set $\mathcal{Q}$. In this work, we focus on five types of demand uncertainty sets used in the work of Subramanyam, Repoussis, and Gounaris (2018), namely the cardinality-constrained set ($\mathcal{Q}_G$), the budget set ($\mathcal{Q}_B$), the factor model set ($\mathcal{Q}_F$), the discrete set ($\mathcal{Q}_D$), and the ellipsoidal set ($\mathcal{Q}_E$). These sets are defined as follows:

$$\mathcal{Q}_G := \left\{ q : q_i = q_i^0 + \xi_i \hat{q}_i, \xi_i \in [0,1], \forall i \in V_c, \sum_{i=1}^{n} \xi_i \leq \Gamma \right\},$$

$$\mathcal{Q}_B := \left\{ q \in \left[\underline{q}, \overline{q}\right] : \sum_{i \in B_l} q_i \leq b_l, \forall l \in \{1,2,...,L\} \right\},$$

$$\mathcal{Q}_F := \left\{ q \in \mathbb{R}^n : q = q^0 + \Psi\xi, \xi \in [-1,1]^F, |e^T\xi| \leq \beta F \right\},$$

$$\mathcal{Q}_D := \text{conv}\left\{ q^{(j)} : j = 1,2,...,D \right\},$$

$$\mathcal{Q}_E := \left\{ q \in \mathbb{R}^n : (q-q^0)^\top \Sigma^{-1} (q-q^0) \leq 1 \right\}.$$

Note that $\mathcal{Q}_G, \mathcal{Q}_B, \mathcal{Q}_F$, and $\mathcal{Q}_D$ are polyhedral sets, while $\mathcal{Q}_E$ represents an ellipsoidal set. Readers are referred to Subramanyam, Repoussis, and Gounaris (2018) for details and exact notation. The objective in is to design cost-effective routes so that vehicle capacities are respected under any demand realization $q \in \mathcal{Q}$.

3. Solution Approach

Let $q^0 \in \mathcal{Q}$ denote the nominal customer demands. Our algorithm works as follows. We define R as the set of all feasible routes with respect to the nominal demands q^0. We highlight that, in

each route $r \in R$, the vehicle capacity is respected only under q^0, and it may be exceeded by the total demand served in a route under some other admissible realization, $q \in \mathcal{Q} \setminus \{q^0\}$. To immunize routing designs against the pre-defined uncertainty set $\mathcal{Q}$, we adopt the *robust rounded capacity inequalities* (robust RCI) from the work of Gounaris, Wiesemann, and Floudas (2013). Let a binary variable λ_r be 1, if route r is selected in the optimal solution, and 0 otherwise. We propose the set-partitioning model (1) - (5).

$$\underset{\lambda_r}{\text{minimize}} \quad \sum_{r \in R} c_r \lambda_r \tag{1}$$

$$\text{subject to} \quad \sum_{r \in R} \delta_{ir} \lambda_r = 1 \qquad \forall\, i \in V_c \tag{2}$$

$$\sum_{r \in R} \lambda_r \leq K \tag{3}$$

$$\sum_{r \in R} \alpha_{Sr} \lambda_r \geq \left\lceil \frac{1}{Q} \max_{q \in \mathcal{Q}} \sum_{i \in S} q_i \right\rceil \qquad \forall\, S \subseteq V_c \tag{4}$$

$$\lambda_r \in \{0,1\} \qquad \forall\, r \in R \tag{5}$$

The objective (1) is to minimize the total travel cost. Eqs. (2) denote degree constraints, while the fleet size constraint is enforced in (3). Inequalities (4) denote the robust RCI, which ensure robust feasibility of the optimal vehicle route. Note that the set partitioning model (1) - (5) has exponentially many variables and constraints; hence, we resort to a branch-price-and-cut approach to solve it. Specifically, we develop a modified version of the algorithm by Pecin et al. (2017). The pricing subproblem is formulated as a deterministic SPPRC under the nominal demand q^0. When column generation converges, if the linear programming solution is integral, robust RCI are separated exactly; otherwise, a Tabu search procedure (Subramanyam, Repoussis, and Gounaris 2018) is invoked to identify violated capacity constraints. We emphasize that both separation procedures can be conducted efficiently for the five uncertainty sets in Section 3.

4. Computational Results

We consider five uncertainty sets, and we generate respective robust VRP instances from each of 90 deterministic CVRP benchmarks, in a similar fashion as Subramanyam, Repoussis, and Gounaris (2018). We then evaluate our algorithm's performance on these instances under a time limit of 2 hours. For the sake of brevity, we only present the preliminary results for the factor model set $\mathcal{Q}_F$ in Table 1. Note how our algorithm could solve 85 of the 90 instances to guaranteed optimality within the time limit; for comparison, the work of Gounaris, Wiesemann, and Floudas (2013) based on branch-and-cut was only able to solve 47 of them optimally within a considerably longer time limit of 24 hours. To the best of our knowledge, this is the first time in the open literature that robust VRP instances under a factor model uncertainty set (Q_F) are tackled by the branch-price-and-cut approach.

Table 1 Computational results of our algorithm on benchmark instances under $\mathcal{Q}_F$

Class	# instance	# opt.	Avg. time (s)	Avg. gap (%)
A	26	26	11	–
B	23	21	19	1.58
E	11	11	21	–
F	3	2	68	1.47
M	3	1	1,573	1.46
P	24	24	6	–
Total	90	85		

5. Conclusions

In this work, we propose a novel approach based on a branch-price-and-cut algorithm for solving robust vehicle routing problems under various, generic types of demand uncertainty sets. Our preliminary results indicate that the proposed algorithm is competitive against utilizing branch-and-cut, which is currently the state-of-the-art approach in addressing robust VRPs under many of the considered types of uncertainty sets. We emphasize that, in contrast with previous works (e.g., Pessoa et al. (2018) and Munari et al. (2019)), our branch-price-and-cut approach can generically handle a variety of uncertainty sets that are used in the context of robust optimization.

References

Ben-Tal A, El Ghaoui L, Nemirovski A, 2009 *Robust optimization*, volume 28 (Princeton University Press).

Gounaris CE, Wiesemann W, Floudas CA, 2013 *The robust capacitated vehicle routing problem under demand uncertainty. Operations Research* 61(3):677–693.

Lee C, Lee K, Park S, 2012 *Robust vehicle routing problem with deadlines and travel time/demand uncertainty. Journal of the Operational Research Society* 63(9):1294–1306.

Lu D, Gzara F, 2019 *The robust vehicle routing problem with time windows: Solution by branch and price and cut. European Journal of Operational Research* 275(3):925–938.

Munari P, Moreno A, De La Vega J, Alem D, Gondzio J, Morabito R, 2019 *The robust vehicle routing problem with time windows: compact formulation and branch-price-and-cut method. Transportation Science* .

Pecin D, Pessoa A, Poggi M, Uchoa E, 2017 *Improved branch-cut-and-price for capacitated vehicle routing. Mathematical Programming Computation* 9(1):61–100.

Pessoa AA, Poss M, Sadykov R, Vanderbeck F, 2018 *Branch-and-cut-and-price for the robust capacitated vehicle routing problem with knapsack uncertainty* .

Subramanyam A, Repoussis PP, Gounaris CE, 2018 *Robust optimization of a broad class of heterogeneous vehicle routing problems under demand uncertainty. arXiv preprint arXiv:1810.04348* .

Toth P, Vigo D, 2014 *Vehicle routing: problems, methods, and applications* (SIAM).