Optimal redeeming strategy of stock loans under drift uncertainty

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November 21, 2017

Abstract

In practice, one must recognize the inevitable presence of information incompleteness when making decisions. In this paper, we consider the optimal redeeming problem of stock loans under incomplete information which is presented by the uncertainty of the trend of underlying stock (called drift uncertainty). Due to the unavoidable estimating of the trend when making decisions, the HJB equation turns out to be a degenerate parabolic PDE; and hence it is very hard to obtain its regularity by standard approaches, making the problem distinguish from the existing optimal redeeming problems without drift uncertainty. We provide a thorough and delicate probabilistic and functional analysis to the value function to obtain its regularity and the optimal redeeming strategies. The latter is shown to be significantly different when the stock is estimated to be in its bull and bear trend.

Keywords: stock loan; drift uncertainty; optimal stopping; degenerate parabolic variational inequality

Mathematics Subject Classification: 91B70; 35R35; 93E20; 60H30

1 Introduction

The classical way (based on the dynamic programming principle) to solve optimal stopping problems assumes that there is a unique known subjective prior distribution driving

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the underlying process. In the context of the classical Black-Scholes framework, in which the underlying asset process follows the geometric Brownian motion with known drift, the theory of financial optimal stopping problems has been well established; see, e.g., [5], [10], [6], [8], [15], [24], [26], [41], [46], [51], [50], [54].

In practice, on the other hand, when designing and controlling physical or organizational systems, one must recognize the inevitable presence of information incompleteness. For a wide range of applications in areas such as engineering, economics and finance, classical stochastic control theory, which is typically based on a single complete nominal model of system, fails to provide strategies that yield satisfactory performance. People today are becoming increasingly aware of the importance of taking information incompleteness into account when dealing with stochastic control problems. In financial applications, as was pointed by Ekstr and Vaicenavicius [13], it is usually too strong to assume that the drift of the underlying asset is known. To obtain a reasonable precise estimation of the drift one needs very long time series, which sometime are rarely available, especially for an initial public offering stock for which the price history simply does not exist. To address this feature, various models have been proposed in the literature; see, e.g., [16], [21], [49], [38], [40], [39], [36], [57], [42],[58].

Surprisingly, there are only few attempts to investigate financial optimal stopping problems under incomplete information; see, [11], [13], [44]. In this paper, we study the optimal redeeming problem of stock loans under incomplete information. A stock loan is a loan between a client (borrower) and a bank (lender), secured by a stock, which gives the borrower the right to redeem the stock at any time before or on the loan maturity by repaying the lender the principal and a predetermined loan interest rate, or surrendering the stock instead of repaying the loan. Xia and Zhou [46] initiated the theoretical study of stock loan redeeming (or equivalently, pricing) problem under the Black-Scholes framework. Through a probabilistic argument, they obtained a closed-form pricing formula for the standard stock loan for which the dividends are gained by the lender before redemption. Dai and Xu [6] extended Xia and Zhou's work to general stock loans with different ways of dividend distributions through a PDE argument. Cai and Sun [3], and Liang, Nie, and Zhao [23] considered models with jumps. Zhang and Zhou [56] and Prager and Zhang [36] studied models under regime-switching.

In regime-switching models, the current trend and volatility of the underlying stock are known to the borrower, although they may change anytime in the future. By contrast, in this paper, we assume the borrower does not know the current trend of the stock so that she/he has to make decisions based on incomplete information. We choose to model the inherent uncertainty of the trend of the stock (called drift uncertainty) by a two-state random variable representing bull and bear trend respectively. The corresponding

Hamilton-Jacobi-Bellman (HJB) equation turns out to be a degenerate parabolic one, which makes our problem much challenging than the existing ones without drift uncertainty. The degeneracy is essentially due to the presence of the drift uncertainty in the model, thus it can not be removed by change of variable or other standard ways in PDE. This unique feature leads to the failure of applying the PDE argument used in [6] to tackle the present problem. In fact, the regularity of the value function is not good enough to let the HJB equation hold almost everywhere; instead, it holds only in the weak sense or viscosity sense. In this paper, we provide a thorough and delicate probabilistic and functional analysis to the value function to obtain its regularity as well as the optimal redeeming strategies. The optimal redeeming strategies turn out to be significantly different when the stock is estimated to be in its bull and bear trend.

This rest part of this paper is organized as follows. We formulate the optimal redeeming problem of stock loans under drift uncertainty in Section 2. Some preliminary results on continuities of the value function are given in Section 3. We study the boundary cases in Section 4 and the general case in Section 5, respectively. A short remark is put in Section 6.

2 Problem formulation

2.1 The market and the underlying stock

We fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, which represents the financial market. As usual, the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ satisfies the usual conditions, and $\mathbf{P}$ denotes the probability measure. In this probability space, there exists a standard one-dimensional Brownian motion W. The price process of the underlying stock is denoted by $S = (S_t)_{t\geqslant 0}$ which evolves according to the stochastic differential equation (SDE):

$$\mathrm{d}S_t = (\mu - \delta)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t,$$

where the dividend $\delta \geqslant 0$ and the volatility $\sigma > 0$ are two known constants, while the return rate μ is unknown. By the self-similarity property of Brownian motion, without loss of generality, we assume $\sigma = 1$ throughout the paper.

To model the inherent uncertainty of the trend of the underlying stock (called drift uncertainty), we assume μ is independent of the Brownian motion W, and $\mu - \delta$ may only take two possible values a and b that satisfy

$$\Delta := a - b > 0.$$

The stock is said to be in its *bull trend* when $\mu - \delta = a$, and in its *bear trend* when $\mu - \delta = b$.

Remark 2.1. *If $\Delta = 0$, then the drift uncertainty disappears and our financial market reduces to the classical Black-Scholes one.*

2.2 The standard stock loan and the optimal redeeming problem of it

A stock loan is a loan, secured by a stock, which gives the borrower the right to redeem the stock at any time before or on the loan maturity. In this paper, we will only study the standard stock loan which features that the dividends are gained by the lender before redemption. For such a stock loan, when the borrower redeems the stock at time t, she/he has to repay the amount of $Ke^{\gamma t}$ to the lender and get back the stock (but the stock dividend gained up to time t will be left to the lender). Here $K > 0$ represents the loan principle (also called loan to value), and γ represents the loan rate. Such a stock loan has been considered under various different frameworks (see, e.g., [6], [46], and [56]). In practice, the loan rate should be higher than the risk-free interest rate

$$\gamma > r,$$

which is henceforth assumed for the rest of this paper.

This paper studies the borrower's optimal redeeming problem that is finding out the optimal stopping time to achieve

$$\sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbf{E}\left[e^{-r(\tau-t)}\left(S_\tau - Ke^{\gamma\tau}\right)^+ \,\middle|\, \mathcal{F}_t^S\right], \tag{2.1}$$

where $\mathcal{T}_{[t,T]}$ denotes the set of all $\mathbb{F}^S$-stopping times valued in $[t,T]$, and r denotes the risk-free interest rate (or discounting rate of the borrower). Here $\mathbb{F}^S = \{\mathcal{F}_t^S\}_{t\geqslant 0}$ is the natural filtration generated by the stock price process, which is observable to the borrower. If $\gamma = 0$, the problem (2.1) reduces to the optimal redeeming problem of the vanilla American call option under drift uncertainty (see more discussions in [46]). In our model, the trend of the underlying stock is unknown, so this is not a standard optimal stopping problem.

To determine the optimal redeeming strategy, the borrower has to estimate the current trend of the stock first. For this, introduce the *a posteriori probability process* $\pi = (\pi_t)_{t\geqslant 0}$ defined as

$$\pi_t := \mathbf{P}\left(\mu - \delta = a \,\middle|\, \mathcal{F}_t^S\right).$$

Roughly speaking, it estimates the probability that the stock is in its bull trend at time t. We assume $0 < \pi_0 < 1$, otherwise the drift uncertainty disappears.

Sometimes, it is more convenient to consider the log-price process $L = (\log S_t)_{t\geqslant 0}$ which by Itô's lemma follows

$$\mathrm{d}L_t = (\mu - \delta - \tfrac{1}{2})\,\mathrm{d}t + \mathrm{d}W_t.$$

We notice that $\mathbb{F}^S$ is the same as $\mathbb{F}^L$, the filtration generated by L. According to the innovation representation (see, e.g., Proposition 2.30 in [1]), the process

$$\overline{W}_t = L_t - \int_0^t \mathbf{E}\big[\mu - \delta - \tfrac{1}{2} \,\big|\, \mathcal{F}_\nu^L\big]\,\mathrm{d}\nu = L_t - \int_0^t \Big(\Delta\pi_\nu + b - \tfrac{1}{2}\Big)\,\mathrm{d}\nu$$

is a Brownian motion under the (observable) filtration $\mathbb{F}^L$. It then follows

$$\mathrm{d}L_t = \Big(\Delta\pi_t + b - \tfrac{1}{2}\Big)\,\mathrm{d}t + \mathrm{d}\overline{W}_t, \tag{2.2}$$

and applying Itô's lemma yields

$$\mathrm{d}S_t = (\Delta\pi_t + b)\,S_t\,\mathrm{d}t + S_t\,\mathrm{d}\overline{W}_t. \tag{2.3}$$

We notice that $\Delta\pi_t + b \leqslant a$, which will be used frequently in the subsequent analysis without claim. An application of the general Bayes formula (see, e.g., Chapter 7.9 in [29]) and Itô's lemma give

$$\mathrm{d}\pi_t = \Delta\pi_t(1-\pi_t)\,\mathrm{d}\overline{W}_t. \tag{2.4}$$

Therefore, solving the problem (2.1), regarded as an optimal stopping problem of the Markovian processes (2.3) and (2.4), reduces to determining

$$V(s,\pi,t) := \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbf{E}\Big[e^{-r(\tau-t)}\,(S_\tau - Ke^{\gamma\tau})^+ \,\Big|\, S_t = s, \pi_t = \pi\Big] \tag{2.5}$$

for (s,π,t) in the domain

$$\mathcal{A} := (0,+\infty)\times(0,1)\times[0,T).$$

This problem is an optimal stopping problem of an observable Markov process $(S_t,\pi_t)_{t\geqslant 0}$, hence the dynamic programming principle may be applied and one would expect to use the variational inequality techniques in PDE to tackle the corresponding HJB equation. Dai and Xu [6] used this method to study the optimal redeeming problem of stock loans in the case without drift uncertainty. Without drift uncertainty, the HJB equation in [6] is uniformly parabolic; in contrast, in our problem it is degenerate parabolic, which is difficult to obtain its regularity from PDE viewpoint,[1] making our problem much challenging than before. In this paper, we will present a thorough and delicate probabilistic

[1] See [4] for an example of the study of a degenerate parabolic equation.

and functional analysis to the value function to obtain its regularity as well as the optimal redeeming strategies.

Because the obstacle in the problem (2.5) is time variant, it is convenient to introduce the discounted stock process $X_t = e^{-\gamma t} S_t$, which follows

$$\mathrm{d}X_t = \Big(\Delta\pi_t + b - \gamma\Big) X_t \,\mathrm{d}t + X_t \,\mathrm{d}\overline{W}_t. \tag{2.6}$$

If we define

$$u(x,\pi,t) := \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbf{E}\Big[e^{(\gamma-r)(\tau-t)} \left(X_\tau - K\right)^+ \Big| X_t = x, \pi_t = \pi\Big]. \tag{2.7}$$

Then one can easily see that

$$V(s,\pi,t) = e^{\gamma t} u(e^{-\gamma t} s, \pi, t), \quad (s,\pi,t) \in \mathcal{A}. \tag{2.8}$$

Thus solving the problem (2.5) reduces to solving (2.7). From now on, we call $u(x,\pi,t)$ defined in (2.7) the value function, and pay attention to it rather than $V(s,\pi,t)$.

Define the continuation region

$$\mathcal{C} := \{(x,\pi,t) \in \mathcal{A} \mid u(x,\pi,t) > (x-K)^+\}, \tag{2.9}$$

and the redeeming region

$$\mathcal{R} := \{(x,\pi,t) \in \mathcal{A} \mid u(x,\pi,t) = (x-K)^+\}. \tag{2.10}$$

By the general theory of optimal stopping, the optimal redeeming strategy for the problem (2.7) as well as (2.5) is given by hitting time of the redeeming region $\mathcal{R}$, that is,

$$\begin{aligned} \tau^* &= \inf\{t \in [0,T) \mid (X_t, \pi_t, t) \in \mathcal{R}\} \wedge T \\ &= \inf\{t \in [0,T) \mid (e^{-\gamma t} S_t, \pi_t, t) \in \mathcal{R}\} \wedge T. \end{aligned}$$

It is not optimal to redeem the stock in the continuation region $\mathcal{C}$.

The main purpose of this paper reduces to studying the value function $u(x,\pi,t)$ and determining the redeeming region $\mathcal{R}$ and the continuation region $\mathcal{C}$.

In the problem (2.7), it is easily seen by choosing $\tau = T$ that $u > 0$ on $\mathcal{A}$. Consequently,

$$\mathcal{C} \supseteq \{(x,\pi,t) \in \mathcal{A} \mid x \leqslant K\}, \quad \mathcal{R} \subseteq \{(x,\pi,t) \in \mathcal{A} \mid x > K\}. \tag{2.11}$$

Therefore, the the continuation region $\mathcal{C}$ is always non-empty. In contrast, the redeeming region $\mathcal{R}$, depending on the parameters as shown below, can be either empty or non-empty.

3 Preliminaries and continuities

In this section, we study the continuities of the value function defined in (2.7) mainly via probabilistic analysis.

3.1 Preliminaries

We will use the following elementary inequalities frequently in the subsequent analysis.

Lemma 3.1. *For any real-valued functions f, g, and real numbers x, y, we have*

1. $|\sup f - \sup g| \leqslant \sup |f - g|$;
2. $x^+ - y^+ \leqslant |x - y|$; *in particular,* $|(x-K)^+ - (y-K)^+| \leqslant |x-y|$;
3. $|e^x - e^y| \leqslant (e^x + e^y)|x - y|$.

Proof. The last inequality follows from the mean value theorem. The others are easy to check. □

The following result gives an upper bound for the value function. As a consequence, the problem (2.7) is meaningful.

Lemma 3.2. *We have*

$$u(x,\pi,t) \leqslant xe^{(a-r)^+(T-t)}, \quad (x,\pi,t) \in \mathcal{A}.$$

Proof. For any $\tau \in \mathcal{T}_{[t,T]}$, applying Itô's lemma to (2.6) gives

$$X_\tau = X_t e^{\int_t^\tau (\Delta\pi_\nu + b - \gamma - \frac{1}{2})\,\mathrm{d}\nu + \overline{W}_\tau - \overline{W}_t} \leqslant X_t e^{(a-\gamma-\frac{1}{2})(\tau-t) + \overline{W}_\tau - \overline{W}_t},$$

and thus

$$\begin{aligned}
u(x,\pi,t) &= \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbf{E}\Big[e^{(\gamma-r)(\tau-t)}\,(X_\tau - K)^+ \,\Big|\, X_t = x, \pi_t = \pi\Big] \\
&\leqslant \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbf{E}\Big[e^{(\gamma-r)(\tau-t)} X_\tau \,\Big|\, X_t = x, \pi_t = \pi\Big] \\
&\leqslant \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbf{E}\Big[X_t e^{(a-r-\frac{1}{2})(\tau-t) + \overline{W}_\tau - \overline{W}_t} \,\Big|\, X_t = x, \pi_t = \pi\Big] \\
&= \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbf{E}\Big[xe^{(a-r)(\tau-t)} e^{-\frac{1}{2}(\tau-t) + \overline{W}_\tau - \overline{W}_t} \,\Big|\, \pi_t = \pi\Big] \\
&\leqslant \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbf{E}\Big[xe^{(a-r)^+(T-t)} e^{-\frac{1}{2}(\tau-t) + \overline{W}_\tau - \overline{W}_t} \,\Big|\, \pi_t = \pi\Big] \\
&= xe^{(a-r)^+(T-t)}.
\end{aligned}$$

□

The next result states the monotonicity of the value function.

Lemma 3.3. *The value function $u(x,\pi,t)$ on $\mathcal{A}$ is non-decreasing in π, non-increasing in t, non-decreasing and convex in x.*

Proof. Both X_t and π_t are stationary processes, so we have

$$\begin{aligned}
u(x,\pi,t) &= \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbf{E}\left[e^{(\gamma-r)(\tau-t)}\left(X_\tau-K\right)^+\middle|X_t=x,\pi_t=\pi\right]\\
&= \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbf{E}\left[e^{(\gamma-r)(\tau-t)}\left(X_t e^{\int_t^\tau(\Delta\pi_\nu+b-\gamma-\frac12)\,\mathrm{d}\nu+\overline{W}_\tau-\overline{W}_t}-K\right)^+\middle|X_t=x,\pi_t=\pi\right]\\
&= \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbf{E}\left[e^{(\gamma-r)(\tau-t)}\left(x e^{\int_t^\tau(\Delta\pi_\nu+b-\gamma-\frac12)\,\mathrm{d}\nu+\overline{W}_\tau-\overline{W}_t}-K\right)^+\middle|\pi_t=\pi\right]\\
&= \sup_{\tau\in\mathcal{T}_{[0,T-t]}} \mathbf{E}\left[e^{(\gamma-r)\tau}\left(x e^{\int_0^\tau(\Delta\pi_\nu+b-\gamma-\frac12)\,\mathrm{d}\nu+\overline{W}_\tau}-K\right)^+\middle|\pi_0=\pi\right]. \qquad (3.1)
\end{aligned}$$

It follows that $u(x,\pi,t)$ is convex and non-decreasing in x, and non-increasing in t.

Now let us show u is non-decreasing in π. Without loss of generality, we may assume $t=0$. Itô's lemma yields

$$\begin{aligned}
\mathrm{d}\log\left(\frac{\pi_t}{1-\pi_t}\right) &= \mathrm{d}\log\pi_t-\mathrm{d}\log(1-\pi_t)\\
&= \Delta(1-\pi_t)\,\mathrm{d}\overline{W}_t-\tfrac12\Delta^2(1-\pi_t)^2\,\mathrm{d}t+\Delta\pi_t\,\mathrm{d}\overline{W}_t+\tfrac12\Delta^2\pi_t^2\,\mathrm{d}t\\
&= \Delta^2\left(\pi_t-\tfrac12\right)\mathrm{d}t+\Delta\,\mathrm{d}\overline{W}_t,
\end{aligned}$$

so that

$$\log\left(\frac{\pi_t}{1-\pi_t}\right)=\log\left(\frac{\pi_0}{1-\pi_0}\right)+\Delta^2\int_0^t\left(\pi_\nu-\tfrac12\right)\mathrm{d}\nu+\Delta\overline{W}_t.$$

If π_t and π'_t are the solutions of (2.4) with initial values $\pi_0>\pi'_0$ (both in $(0,1)$), respectively. Then $\tau=\inf\{t\geqslant 0\mid\pi_t\leqslant\pi'_t\}>0$. If $\tau(\omega)$ is finite, then by continuity we have $\pi_\tau(\omega)=\pi'_\tau(\omega)\in(0,1)$ and thus

$$\begin{aligned}
0 &= \log\left(\frac{\pi_\tau}{1-\pi_\tau}\right)-\log\left(\frac{\pi'_\tau}{1-\pi'_\tau}\right)\\
&= \log\left(\frac{\pi_0}{1-\pi_0}\right)-\log\left(\frac{\pi'_0}{1-\pi'_0}\right)+\Delta^2\int_0^\tau(\pi_\nu-\pi'_\nu)\,\mathrm{d}\nu>0,
\end{aligned}$$

a contradiction. So $\tau(\omega)=+\infty$ and consequently $\pi_t(\omega)>\pi'_t(\omega)$ for all t. Therefore π_t is non-decreasing ω-wisely with respect to the initial value π_0, so from the last expression in (3.1) we see that u is non-decreasing in π. □

Lemma 3.4. *We have, uniformly for all $0\leqslant y\leqslant x<\infty$, $\pi\in[0,1]$ and $t\in[0,T]$, that*

$$0\leqslant u(x,\pi,t)-u(y,\pi,t)\leqslant(x-y)e^{(a-r)^+(T-t)}. \qquad (3.2)$$

Especially, if $r\geqslant a$, then

$$0\leqslant u(x,\pi,t)-u(y,\pi,t)\leqslant x-y. \qquad (3.3)$$

Proof. Applying Lemma 3.1 to the last expression in (3.1), we have

$$\begin{aligned}
|u(x,\pi,t)-u(y,\pi,t)| &\leqslant |x-y| \sup_{\tau\in\mathcal{T}_{[0,T-t]}} \mathbf{E}\left[e^{(\gamma-r)\tau}e^{\int_0^\tau(\Delta\pi_\nu+b-\gamma-\frac{1}{2})\,\mathrm{d}\nu+\overline{W}_\tau}\middle|\pi_0=\pi\right]\\
&\leqslant |x-y| \sup_{\tau\in\mathcal{T}_{[0,T-t]}} \mathbf{E}\left[e^{(a-r)\tau}e^{-\frac{1}{2}\tau+\overline{W}_\tau}\middle|\pi_0=\pi\right]\\
&\leqslant |x-y|e^{(a-r)^+(T-t)}.
\end{aligned}$$

□

Lemma 3.5. *We have, uniformly for all $x\in(0,\infty)$, $0\leqslant\varpi\leqslant\pi\leqslant 1$ and $t\in[0,T]$, that*

$$0\leqslant u(x,\pi,t)-u(x,\varpi,t)\leqslant C\Delta x(T-t)(\pi-\varpi), \tag{3.4}$$

for some constant $C>0$.

Proof. Let π_t and π'_t be the solutions of (2.4) with initial values $\pi\geqslant\varpi$, respectively. And denote

$$Y_t=\int_0^t(\Delta\pi_\nu+b-\gamma-\tfrac{1}{2})\,\mathrm{d}\nu+\overline{W}_t\leqslant(a-\gamma+\tfrac{1}{2})^+t+\overline{W}_t-t,$$

and

$$Y'_t=\int_0^t(\Delta\pi'_\nu+b-\gamma-\tfrac{1}{2})\,\mathrm{d}\nu+\overline{W}_t\leqslant(a-\gamma+\tfrac{1}{2})^+t+\overline{W}_t-t.$$

Then

$$\sup_{s\leqslant T}|Y_s-Y'_s|\leqslant\Delta T\sup_{s\leqslant T}|\pi_s-\pi'_s|.$$

Similar as before, applying Lemma 3.1 to the last expression in (3.1) and using Cauthy's inequality, we have

$$\begin{aligned}
|u(x,\pi,t)-u(x,\varpi,t)| &\leqslant x\sup_{\tau\in\mathcal{T}_{[0,T-t]}}\mathbf{E}\left[e^{(\gamma-r)\tau}\left|e^{Y_\tau}-e^{Y'_\tau}\right|\middle|\pi_0=\pi,\pi'_0=\varpi\right]\\
&\leqslant xe^{(\gamma-r)(T-t)}\sup_{\tau\in\mathcal{T}_{[0,T-t]}}\mathbf{E}\left[\left|e^{Y_\tau}-e^{Y'_\tau}\right|\middle|\pi_0=\pi,\pi'_0=\varpi\right]\\
&\leqslant xe^{(\gamma-r)(T-t)}\sup_{\tau\in\mathcal{T}_{[0,T-t]}}\mathbf{E}\left[\left(e^{Y_\tau}+e^{Y'_\tau}\right)|Y_\tau-Y'_\tau|\middle|\pi_0=\pi,\pi'_0=\varpi\right]\\
&\leqslant xe^{(\gamma-r)(T-t)}\sup_{\tau\in\mathcal{T}_{[0,T-t]}}\mathbf{E}\left[2e^{(a-\gamma+\frac{1}{2})^+(T-t)+\overline{W}_\tau-\tau}|Y_\tau-Y'_\tau|\middle|\pi_0=\pi,\pi'_0=\varpi\right]\\
&\leqslant 2xe^{(\gamma-r+(a-\gamma+\frac{1}{2})^+)(T-t)}\sup_{\tau\in\mathcal{T}_{[0,T-t]}}\left(\mathbf{E}\left[e^{2\overline{W}_\tau-2\tau}\middle|\pi_0=\pi,\pi'_0=\varpi\right]\right)^{1/2}\\
&\quad\times\sup_{\tau\in\mathcal{T}_{[0,T-t]}}\left(\mathbf{E}\left[(Y_\tau-Y'_\tau)^2\middle|\pi_0=\pi,\pi'_0=\varpi\right]\right)^{1/2}\\
&\leqslant Cx\left(\Delta^2(T-t)^2\mathbf{E}\left[\sup_{s\leqslant T-t}(\pi_s-\pi'_s)^2\middle|\pi_0=\pi,\pi'_0=\varpi\right]\right)^{1/2}\\
&\leqslant Cx\Delta(T-t)(\pi-\varpi).
\end{aligned}$$

where the last inequality is due to (1.19), p.25, in [35]. Here and hereafter the implied constant C may vary on each appearance. □

Lemma 3.6. *We have, uniformly for $x \in (0,\infty)$, $\pi \in [0,1]$ and $0 \leqslant s \leqslant t \leqslant T$, that*

$$0 \leqslant u(x,\pi,s) - u(x,\pi,t) \leqslant Cx(t-s)^{\frac{1}{2}}, \tag{3.5}$$

for some constant $C > 0$.

Proof. Denote

$$Y_t = \int_0^t (\Delta\pi_\nu + b - r - \tfrac{1}{2})\,\mathrm{d}\nu + \overline{W}_t \leqslant (a - r - \tfrac{1}{2})t + \overline{W}_t.$$

For any $\tau \in \mathcal{T}_{[0,T-s]}$, set $\tau' = \tau \wedge (T-t) \in \mathcal{T}_{[0,T-t]}$. Then $0 \leqslant \tau - \tau' \leqslant t - s$ and

$$e^{(\gamma-r)\tau}(X_\tau - K)^+ = (e^{(\gamma-r)\tau}X_\tau - Ke^{(\gamma-r)\tau})^+ \leqslant (X_0 e^{Y_\tau} - Ke^{(\gamma-r)\tau'})^+.$$

Using Lemma 3.1, we have

$$\begin{aligned}
0 &\leqslant u(x,\pi,s) - u(x,\pi,t) \\
&\leqslant \sup_{\tau\in\mathcal{T}_{[0,T-s]}} \mathbf{E}\Big[e^{(\gamma-r)\tau}(X_\tau - K)^+ - e^{(\gamma-r)\tau'}(X_{\tau'} - K)^+\Big|X_0 = x, \pi_0 = \pi\Big] \\
&\leqslant \sup_{\tau\in\mathcal{T}_{[0,T-s]}} \mathbf{E}\Big[(xe^{Y_\tau} - Ke^{(\gamma-r)\tau'})^+ - (xe^{Y_{\tau'}} - Ke^{(\gamma-r)\tau'})^+\Big|\pi_0 = \pi\Big] \\
&\leqslant x \sup_{\tau\in\mathcal{T}_{[0,T-s]}} \mathbf{E}\Big[\Big|e^{Y_\tau} - e^{Y_{\tau'}}\Big|\Big|\pi_0 = \pi\Big] \\
&\leqslant x \sup_{\tau\in\mathcal{T}_{[0,T-s]}} \mathbf{E}\Big[\Big(e^{Y_\tau} + e^{Y_{\tau'}}\Big)\Big|Y_\tau - Y_{\tau'}\Big|\Big|\pi_0 = \pi\Big] \\
&\leqslant x \sup_{\tau\in\mathcal{T}_{[0,T-s]}} \Big(\mathbf{E}\Big[\Big(e^{Y_\tau} + e^{Y_{\tau'}}\Big)^2\Big|\pi_0 = \pi\Big]\Big)^{1/2} \sup_{\tau\in\mathcal{T}_{[0,T-s]}} \Big(\mathbf{E}\Big[\Big(Y_\tau - Y_{\tau'}\Big)^2\Big|\pi_0 = \pi\Big]\Big)^{1/2} \\
&\leqslant Cx\left(\sup_{\tau\in\mathcal{T}_{[0,T-s]}} \mathbf{E}\left[\left(\int_{\tau'}^{\tau}(\Delta\pi_\nu + b - r - \tfrac{1}{2})\,\mathrm{d}\nu + \overline{W}_\tau - \overline{W}_{\tau'}\right)^2\Big|\pi_0 = \pi\right]\right)^{1/2} \\
&\leqslant Cx\left(\sup_{\tau\in\mathcal{T}_{[0,T-s]}} \mathbf{E}\Big[\Big(C(t-s) + |\overline{W}_\tau - \overline{W}_{\tau'}|\Big)^2\Big]\right)^{1/2} \\
&\leqslant Cx\left((t-s)^2 + \sup_{\tau\in\mathcal{T}_{[0,T-s]}} \mathbf{E}\Big[\Big(\overline{W}_\tau - \overline{W}_{\tau'}\Big)^2 - (\tau - \tau')\Big] + \sup_{\tau\in\mathcal{T}_{[0,T-s]}} \mathbf{E}[\tau - \tau']\right)^{1/2} \\
&\leqslant Cx(t-s)^{1/2}.
\end{aligned}$$

□

Since monotonic function is differentiable almost everywhere, the above lemmas lead to

Corollary 3.7. *The value function $u(x,\pi,t)$ is continuous in $\overline{\mathcal{A}} := [0,\infty)\times[0,1]\times[0,T]$. The partial derivatives u_x, u_π and u_t exist almost everywhere in $\mathcal{A}$; and uniformly in $\overline{\mathcal{A}}$, we have*

$$0 \leqslant u_x \leqslant C,$$
$$0 \leqslant u_\pi \leqslant Cx,$$
$$u_t \leqslant 0,$$

for some constant $C > 0$. Moreover, $|u_t|$ is integrable in any bounded domain of $\mathcal{A}$.

4 Variational inequality, and the boundary cases

4.1 Degenerate variational inequality

Applying the dynamic programming principle, we see that the value function satisfies the variational inequality

$$\begin{cases} \min\left\{-\mathcal{L}u,\ u-(x-K)^+\right\} = 0, \quad (x,\pi,t)\in\mathcal{A}; \\ u(x,\pi,T) = (x-K)^+, \end{cases} \tag{4.1}$$

where

$$\begin{aligned}\mathcal{L}u := u_t + \tfrac{1}{2}x^2u_{xx} + \tfrac{1}{2}\Delta^2\pi^2(1-\pi)^2u_{\pi\pi} + \Delta\pi(1-\pi)xu_{x\pi} \\ + (\Delta\pi + b - \gamma)xu_x + (\gamma - r)u.\end{aligned}$$

We will thoroughly study the free boundary of this variational inequality in the subsequent sections.

The operator $\mathcal{L}$ in (4.1) is a parabolic operator, which is degenerate in the whole domain $\mathcal{A}$. The regularity of $u(x,\pi,t)$ is not good enough to let (4.1) hold almost everywhere. It holds only in the weak sense or viscosity sense. The definition of weak solution can be found in [14] and [4], and the definition of viscosity solution can be found in [53].

Usual way to get the weak solution of (4.1) is regularization. That is to add a term $\varepsilon u_{\pi\pi}$ to $\mathcal{L}u$ for $\varepsilon > 0$. One can first show there is a "good" u_ε that solves

$$\begin{cases} \min\left\{-(\mathcal{L}+\varepsilon\partial_{\pi\pi})u_\varepsilon,\ u_\varepsilon-(x-K)^+\right\} = 0, \quad (x,\pi,t)\in\mathcal{A}; \\ u_\varepsilon(x,\pi,T) = (x-K)^+, \end{cases} \tag{4.2}$$

where suitable boundaries conditions should be put on the boundaries $\pi = 0, 1$. Then show the limit $\lim_{\varepsilon\to 0+} u_\varepsilon$ solves (4.1) in the weak sense. In doing so, the difficulty is to get estimates, just like in Lemmas 3.2-3.6, for u_ε uniformly with respect to ε. This is a long way, see [4], where the regularity of the value function is the same as in Lemmas 3.2-3.6 in the present paper.

Remark 4.1 (On boundary conditions). *Because the operator $\mathcal{L}$ in (4.1) is degenerate, according to Fichera Theorem [33], boundary conditions for (4.1) must not be put on $x = 0$, $\pi = 0$ or $\pi = 1$. But, on the other hand, we can determine the boundary values in priority.*

- *Let $x = 0$ in (4.1), then $u(0, \pi, t)$ is the solution of the system*

$$\begin{cases} \min\left\{-u_t(0,\pi,t) - \frac{1}{2}\Delta^2\pi^2(1-\pi)^2 u_{\pi\pi}(0,\pi,t) + ru(0,\pi,t),\ u(0,\pi,t)\right\} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\pi,t) \in (0,1)\times[0,T); \\ u(0,\pi,T) = 0. \end{cases}$$

  Obviously, $u(0, \pi, t) \equiv 0$ is the unique solution of this problem.

- *Let $\pi = 0$ in (4.1), then $u(x, 0, t)$ is the solution of the system*

$$\begin{cases} \min\left\{-u_t(x,0,t) - \frac{1}{2}x^2 u_{xx}(x,0,t) - (b-\gamma)xu_x(x,0,t) + ru(x,0,t), \right. \\ \qquad\qquad \left. u(x,0,t) - (x-K)^+\right\} = 0, \quad (x,t) \in (0,+\infty)\times[0,T); \\ u(x,0,T) = (x-K)^+. \end{cases} \tag{4.3}$$

- *Let $\pi = 1$ in (4.1), then $u(x, 1, t)$ is the solution of the system*

$$\begin{cases} \min\left\{-u_t(x,1,t) - \frac{1}{2}x^2 u_{xx}(x,1,t) - (a-\gamma)xu_x(x,1,t) + ru(x,1,t), \right. \\ \qquad\qquad \left. u(x,1,t) - (x-K)^+\right\} = 0, \quad (x,t) \in (0,+\infty)\times[0,T); \\ u(x,1,T) = (x-K)^+. \end{cases} \tag{4.4}$$

The problems (4.3) and (4.4) are free of drift uncertainty, and they have been thoroughly studied by Dai and Xu [6] under the assumptions that $b \leqslant r$ and $a \leqslant r$, respectively.

4.2 The boundary cases: $\pi = 0, 1$

Before studying (4.1), it is necessary to know the situation on the boundaries $\pi = 0, 1$. These two cases are similar: one only needs to interchange a and b. So we assume $\pi = 0$ in the rest part of this section. As a consequence, there is no drift uncertainty and the model reduces to the classical Black-Scholes one.

Let $\pi = 0$ in (4.1), then $u_0(x, t) := u(x, 0, t)$ is the solution of

$$\begin{cases} \min\left\{-\mathcal{L}^0 u_0, \quad u_0 - (x-K)^+\right\} = 0, \quad (x,t) \in \mathcal{A}_0; \\ u_0(x,T) = (x-K)^+, \end{cases} \tag{4.5}$$

where

$$\begin{aligned} \mathcal{L}^0 u &:= u_t + \tfrac{1}{2}x^2 u_{xx} + (b-\gamma)xu_x + (\gamma - r)u, \\ \mathcal{A}_0 &:= (0,+\infty)\times[0,T). \end{aligned}$$

Remark 4.2. *According to the calculation in (5.9) below, there is no free boundary if* $b \geqslant \gamma$.

As $\gamma > r$, we will only need to consider the following two cases: $\gamma > r > b$, and $\gamma > b \geqslant r$.

In [6], the problem (4.5) was thoroughly studied in the case of $r > b$. Especially, they have shown there exists a non-increasing redeeming boundary $X_0(t)$ with the terminal value

$$X_0(T) := \lim_{t\to T} X_0(t) = \max\left\{K,\ \frac{r-\gamma}{r-b}K\right\}.$$

Now we deal with the case of $\gamma > b \geqslant r$. Define two free boundaries for $t \in [0, T)$,

$$X_1(t) := \inf\{x > 0 \mid u_0(x,t) = (x-K)^+\},$$
$$X_2(t) := \sup\{x > 0 \mid u_0(x,t) = (x-K)^+\},$$

with the convention that $X_1(T) := \lim_{t\to T} X_1(t)$, $X_2(T) := \lim_{t\to T} X_2(t)$, and $\sup\emptyset = +\infty$. We say $X_1(t)$ disappears if the set $\{x > 0 \mid u_0(x,t) = (x-K)^+\}$ is empty; $X_2(t)$ disappears if it is $+\infty$. Clearly, we have

$$\{(x,t) \in \mathcal{A}_0 \mid u_0 = (x-K)^+\} = \{(x,t) \in \mathcal{A}_0 \mid X_1(t) \leqslant x \leqslant X_2(t)\}.$$

Lemma 4.1. *If* $\gamma > b > r$, *then* $X_1(t)$ *is strictly decreasing and* $X_2(t)$ *is strictly increasing with the the terminal values* $X_1(T) = K$ *and* $X_2(T) = \frac{\gamma-r}{b-r}K$.

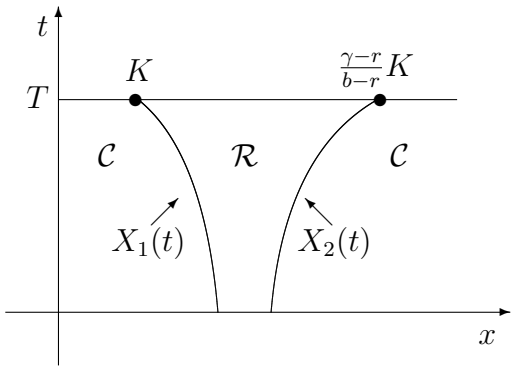


Figure 1: $\pi = 0$ and $\gamma > b > r$.

Proof. If

$$-\mathcal{L}^0(x-K) = (r-b)x + (\gamma-r)K \geqslant 0,$$

then $x \leqslant \frac{\gamma-r}{b-r}K$, so we have

$$\{(x,t) \in \mathcal{A}_0 \mid u_0 = (x-K)^+\} \subseteq \{(x,t) \in \mathcal{A}_0 \mid K \leqslant x \leqslant \tfrac{\gamma-r}{b-r}K\}.$$

Since u_0 is non-increasing in t, we see that $X_1(t)$ is non-increasing and $X_2(t)$ is non-decreasing. Moreover, we can prove that, by Hopf lemma in PDE, anyone of them, if exists, is strictly monotonic.

Now let us prove $X_1(T) = K$. First we have $X_1(T) \geqslant K$. If $X_1(T) > K$, then for sufficient small $\varepsilon > 0$ and $(x,t) \in (K, X_1(T)) \times (T-\varepsilon, T)$,

$$-\mathcal{L}^0 u_0 = 0, \quad u_0(x,T) = x - K,$$

hence

$$\partial_t u_0(x,t) = -(b-r)x + (\gamma - r)K,$$

which is positive at $(x,t) = (K,T)$, contradicting $\partial_t u_0 \leqslant 0$. So $X_1(T) = K$. In the same way $X_2(T) = \frac{\gamma - r}{b-r} K$ can be proved. □

Proposition 4.2. *In the case of $\gamma > b > r$, we have*

- *If $\gamma \geqslant b + \frac{1}{2} + \sqrt{2b - 2r}$, then the two redeeming boundaries $X_1(t)$ and $X_2(t)$ exist and*

$$X_1(t) < \left(1 + \tfrac{1}{\sqrt{2b-2r}}\right) K < X_2(t) \tag{4.6}$$

for all $0 \leqslant t \leqslant T$.

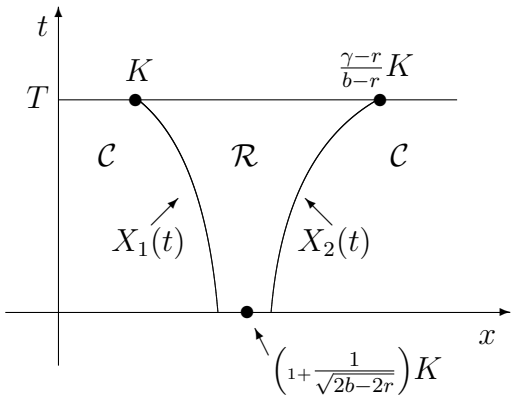


Figure 2: $\pi = 0$, $\gamma \geqslant b + \frac{1}{2} + \sqrt{2b - 2r}$ and $b > r$.

- *If $\gamma < b + \frac{1}{2} + \sqrt{2b - 2r}$, then there exists $\ell < T$ such that the two redeeming boundaries $X_1(t)$ and $X_2(t)$ exist for $T - \ell \leqslant t \leqslant T$, intersect at $T - \ell$; and both of them disappear for $t < T - \ell$.*[2]

[2]Here, ℓ does not depend on T. And an upper bound for ℓ can be given explicitly as shown in the proof.

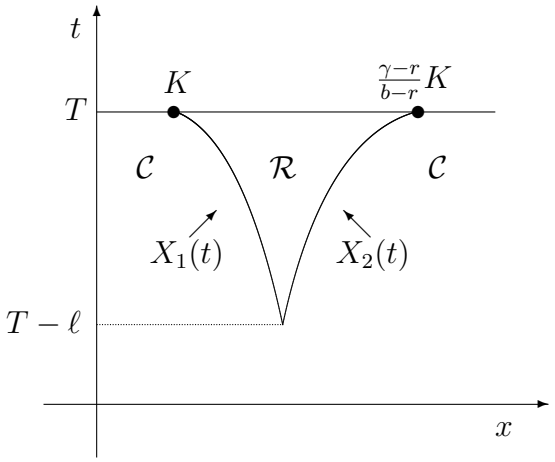


Figure 3: $\pi = 0$ and $r < b < \gamma < b + \frac{1}{2} + \sqrt{2b - 2r}$.

We would like to give a financial interpretation for this result before giving its proof.

In the first case, the loan rate is relatively high. The borrower should redeem the stock when its price increases to the redeeming boundary $X_1(t)$ because the loan rate is too high to wait. While if the stock price is higher than $X_2(t)$, it seems that one should wait for its price down to $X_2(t)$ to redeem the stock. In fact, however, no client shall sign such a stock loan with bank at the very beginning, because the principle is too few (or the loan rate is too high) compared to the value of the collateral (namely the stock).

In the second case, the loan rate is too low such that the stock loan can be almost regarded as an American call option (which shall not be exercised before maturity by Merton's theorem). Hence the borrower of the stock loan should wait if the time to maturity is very far.

Proof. Set $\gamma_0 = b + \frac{1}{2} + \sqrt{2b - 2r}$.

- Suppose $\gamma \geqslant \gamma_0$. Set $\lambda_0 = 1 + \sqrt{2b - 2r}$, $x_0 = \frac{\lambda_0}{\lambda_0 - 1} K$ and define

$$v(x) = \left(\frac{x}{\lambda_0}\right)^{\lambda_0} \left(\frac{\lambda_0 - 1}{K}\right)^{\lambda_0 - 1}, \qquad x > 0.$$

  From $v(x_0) = \frac{1}{\lambda_0 - 1} K = x_0 - K$, $v'(x_0) = 1$, and the strictly convexity of v, we conclude

$$v(x) = (x - K)^+, \quad \text{if } x = x_0;$$
$$v(x) > (x - K)^+, \quad \text{if } x \neq x_0.$$

Moreover,

$$\begin{aligned}
-\mathcal{L}^0 v &= -v_t - \tfrac{1}{2}x^2 v_{xx} - (b-\gamma)xv_x - (\gamma - r)v \\
&= (-\tfrac{1}{2}\lambda_0(\lambda_0 - 1) - (b-\gamma)\lambda_0 - (\gamma - r))v \\
&= (-\tfrac{1}{2}\lambda_0(\lambda_0 - 1) - b\lambda_0 + \gamma(\lambda_0 - 1) + r)v \\
&\geqslant (-\tfrac{1}{2}\lambda_0(\lambda_0 - 1) - b\lambda_0 + \gamma_0(\lambda_0 - 1) + r)v \\
&= (-\tfrac{1}{2}\lambda_0(\lambda_0 - 1) - b\lambda_0 + (\lambda_0 + b - \tfrac{1}{2})(\lambda_0 - 1) + r)v \\
&= (\tfrac{1}{2}(\lambda_0 - 1)^2 - b + r)v \\
&= 0.
\end{aligned}$$

Therefore, v is a supper solution of (4.5), and hence

$$(x-K)^+ \leqslant u_0(x,t) \leqslant v(x), \quad (x,t) \in \mathcal{A},$$

from which we get $u_0(x_0,t) = (x_0 - K)^+$ for $0 \leqslant t \leqslant T$. Notice

$$x_0 = \tfrac{\lambda_0}{\lambda_0 - 1}K = \left(1 + \tfrac{1}{\sqrt{2b-2r}}\right)K,$$

so (4.6) follows from the strictly monotonicity of $X_1(t)$ and $X_2(t)$.

- Now suppose $\gamma < \gamma_0$. We note that by (3.1)

$$\begin{aligned}
u_0(x,t) &= \sup_{\tau \in \mathcal{T}_{[0,T-t]}} \mathbf{E}\left[e^{(\gamma-r)\tau}\left(xe^{\int_0^\tau (\Delta\pi_\nu + b - \gamma - \frac{1}{2})\,\mathrm{d}\nu + \overline{W}_\tau} - K\right)^+ \middle| \pi_0 = 0\right] \\
&= \sup_{\tau \in \mathcal{T}_{[0,T-t]}} \mathbf{E}\left[e^{(\gamma-r)\tau}\left(xe^{(b-\gamma-\frac{1}{2})\tau + \overline{W}_\tau} - K\right)^+\right] \\
&\geqslant \mathbf{E}\left[e^{(\gamma-r)(T-t)}\left(xe^{(b-\gamma-\frac{1}{2})(T-t) + \overline{W}_{T-t}} - K\right)^+\right] \\
&= xe^{(b-r)(T-t)}N\left(\tfrac{\log x - \log K - (\gamma - b - \frac{1}{2})(T-t)}{\sqrt{T-t}}\right) \\
&\quad - Ke^{(\gamma-r)(T-t)}N\left(\tfrac{\log x - \log K - (\gamma - b + \frac{1}{2})(T-t)}{\sqrt{T-t}}\right) =: g(x,t),
\end{aligned}$$

where N stands for the standard normal distribution. We note that for $x \geqslant K$,

$$\begin{aligned}
\partial_x g(x,t) &= e^{(b-r)(T-t)}N\left(\tfrac{\log x - \log K - (\gamma - b - \frac{1}{2})(T-t)}{\sqrt{T-t}}\right) \\
&\geqslant e^{(b-r)(T-t)}N\left(-(\gamma - b - \tfrac{1}{2})\sqrt{T-t}\right).
\end{aligned}$$

If we can prove the right hand side in above is $\geqslant 1$ for sufficiently large $T-t$, then $g(x,t) > x - K$ for all $x \geqslant K$ as $g(K,t) > 0$ by definition. Consequently, we deduce $u_0(x,t) > (x-K)^+$ for sufficiently large $T-t$ and the claim follows.

  - If $\gamma \leqslant b + \frac{1}{2}$. Then $\partial_x g(x,t) \geqslant e^{(b-r)(T-t)}N(0) \geqslant 1$ for sufficiently large $T-t$.

– If $b+\frac{1}{2}<\gamma<\gamma_0$. Using the well-known inequality

$$N(-x)\geqslant \frac{1}{\sqrt{2\pi}}\frac{x}{(x^2+1)}e^{-\frac{1}{2}x^2},\quad x\geqslant 0,$$

we have for sufficiently large $T-t$,

$$\begin{aligned}\partial_x g(x,t)&\geqslant e^{(b-r)(T-t)}N\left(-(\gamma-b-\tfrac{1}{2})\sqrt{T-t}\right)\\&\geqslant \frac{1}{\sqrt{2\pi}}\frac{(\gamma-b-\frac{1}{2})\sqrt{T-t}}{(\gamma-b-\frac{1}{2})^2(T-t)+1}e^{\left(b-r-\frac{1}{2}(\gamma-b-\frac{1}{2})^2\right)(T-t)}\geqslant 1,\end{aligned}$$

in view of $b-r-\frac{1}{2}(\gamma-b-\frac{1}{2})^2>b-r-\frac{1}{2}(\gamma_0-b-\frac{1}{2})^2=0$. □

Proposition 4.3. *In the case of $\gamma>b=r$, there is only one non-increasing redeeming boundary $X_1(t)$ with the terminal value $X_1(T)=K$. Moreover, the redeeming boundary satisfies the following properties.*

- *If $\gamma>r+\frac{1}{2}$, then $X_1(t)$ exists and $X_1(t)\leqslant \frac{2(\gamma-r)}{2(\gamma-r)-1}K$ for $0\leqslant t\leqslant T$.*[3]

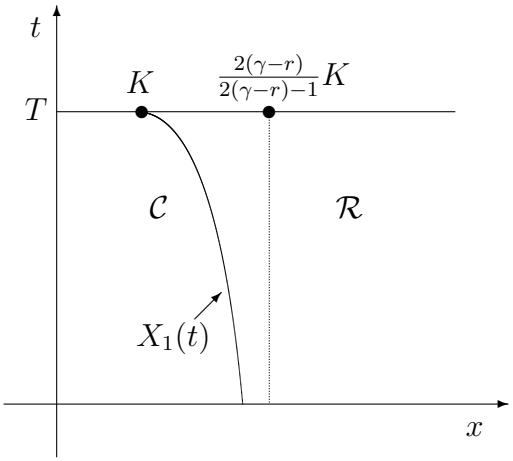


Figure 4: $\pi=0$, $b=r$ and $\gamma>r+\frac{1}{2}$.

- *If $\gamma\leqslant r+\frac{1}{2}$, then $\lim_{T-t\to+\infty}X_1(T-t)=+\infty$.*[4]

[3] In this case, one can prove $\lim_{T-t\to+\infty}X_1(T-t)=\frac{2(\gamma-r)}{2(\gamma-r)-1}K$. Due to space limit, we leave the proof to the interested readers.

[4] We suspect that X_1 disappears for some finite time t in this case, but cannot prove it.

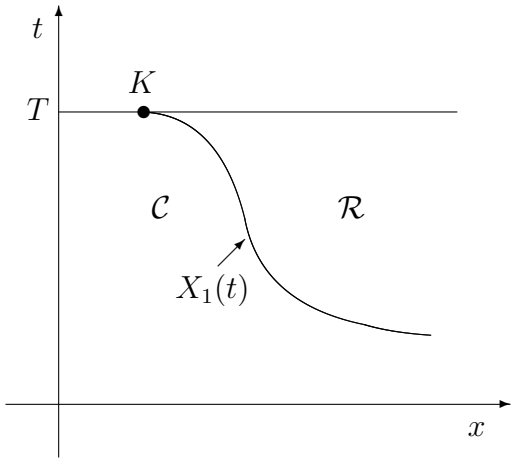


Figure 5: $\pi = 0$ and $b = r < \gamma \leqslant r + \frac{1}{2}$.

Proof. Assume $\gamma > b = r$. By Lemma 3.4, we have

$$0 \leqslant u_0(x,t) - u_0(y,t) \leqslant x - y, \quad x \geqslant y \geqslant 0,$$

that is

$$-K \leqslant u_0(x,t) - x \leqslant u_0(y,t) - y.$$

Therefore, if $u_0(y,t) = y - K$ for some y, then $u_0(x,t) = x - K$ for any $x \geqslant y$. Hence the free boundary $X_2(t)$ disappears.

- Suppose $\gamma > r + \frac{1}{2}$. Let $\lambda_1 = 2(\gamma - r) > 1$, $x_1 = \frac{\lambda_1}{\lambda_1 - 1} K$ and

$$w(x) = \begin{cases} \left(\frac{x}{\lambda_1}\right)^{\lambda_1} \left(\frac{\lambda_1 - 1}{K}\right)^{\lambda_1 - 1}, & 0 < x < x_1; \\ (x - K)^+, & x \geqslant x_1. \end{cases}$$

  It can be seen that $w(x)$ is a supersolution of (4.5), hence

$$u_0(x,t) = (x - K)^+, \quad x \geqslant x_1 = \frac{2(\gamma - r)}{2(\gamma - r) - 1} K.$$

  The claim follows immediately.

- Suppose $\gamma \leqslant r + \frac{1}{2}$. Since x is a supersolution of (4.5) with $b = r$, $u_0(x,t) \leqslant x$. Let

$$u_0^\infty(x) = \lim_{T - t \to +\infty} u_0(x,t) \leqslant x.$$

  Then it solves the corresponding stationary problem of (4.5)

$$\min\left\{ -\tfrac{1}{2} x^2 \partial_{xx} u_0^\infty(x) + (\gamma - r)(x \partial_x u_0^\infty(x) - u_0^\infty(x)), \quad u_0^\infty(x) - (x - K)^+ \right\} = 0, \quad x > 0.$$

  Let

$$x_0 = \lim_{T - t \to +\infty} X_1(T - t) \in [K, +\infty].$$

Then u_0^∞ and x_0 satisfy

$$\begin{cases} -\frac{1}{2}x^2\partial_{xx}u_0^\infty(x) + (\gamma - r)(x\partial_x u_0^\infty(x) - u_0^\infty(x)) = 0, & 0 < x < x_0; \\ (x-K)^+ \leqslant u_0^\infty(x) \leqslant x, & x > 0. \end{cases} \tag{4.7}$$

Moreover,

$$\text{if } x_0 < +\infty, \text{ then } u_0^\infty(x_0) = x_0 - K, \text{ and } \partial_x u_0^\infty(x_0) = 1. \tag{4.8}$$

The general solution of the equation (4.7) is

$$u_0^\infty(x) = \begin{cases} C_1 x \ln x + C_2 x, & \text{if } \gamma = r + \frac{1}{2}; \\ C_1 x^{2(\gamma - r)} + C_2 x, & \text{if } \gamma < r + \frac{1}{2}; \end{cases} \quad 0 < x < x_0.$$

Letting x go to 0 in the condition $\left(1 - \frac{K}{x}\right)^+ \leqslant \frac{u_0^\infty(x)}{x} \leqslant 1$, we obtain $C_1 = 0$. Hence $u_0^\infty(x) = C_2 x$ for $0 < x < x_0$. Consequently, the condition (4.8) cannot be satisfied for any $x_0 < \infty$. So $x_0 = +\infty$, and the claim follows.

The proof is complete. □

5 General case: $0 < \pi < 1$

Now we go back to the general case $0 < \pi < 1$.

We will discuss the behaviors of the redeeming boundaries. From Lemma 3.3 we see that

$$\partial_t \left(u(x,\pi,t) - (x-K)^+\right) \leqslant 0, \quad (x,\pi,t) \in \mathcal{A}; \tag{5.1}$$

$$\partial_\pi \left(u(x,\pi,t) - (x-K)^+\right) \geqslant 0, \quad (x,\pi,t) \in \mathcal{A}. \tag{5.2}$$

Moreover, (3.3) reveals

$$\partial_x \left(u(x,\pi,t) - (x-K)^+\right) \leqslant 0, \quad \text{if } r \geqslant a,\ x \geqslant K,\ (x,\pi,t) \in \mathcal{A}. \tag{5.3}$$

Define, for each fixed $(\pi, t) \in (0,1) \times [0,T)$,

$$X_1(\pi,t) := \min\{x \mid u(x,\pi,t) = (x-K)^+,\ (x,\pi,t) \in \mathcal{A}\}, \tag{5.4}$$

$$X_2(\pi,t) := \max\{x \mid u(x,\pi,t) = (x-K)^+,\ (x,\pi,t) \in \mathcal{A}\}. \tag{5.5}$$

Then

Lemma 5.1. *We have*

$$\mathcal{C} = \{(x, \pi, t) \in \mathcal{A} \mid x < X_1(\pi, t) \text{ or } x > X_2(\pi, t)\}, \tag{5.6}$$

$$\mathcal{R} = \{(x, \pi, t) \in \mathcal{A} \mid X_1(\pi, t) \leqslant x \leqslant X_2(\pi, t)\}. \tag{5.7}$$

Moreover, $X_1(\pi, t)$ is non-increasing in t and non-decreasing in π, while $X_2(\pi, t)$ is non-decreasing in t and non-increasing in π, if they exist; $X_1(\pi, t) > K$.

Proof. For each fixed $(\pi, t) \in (0, 1) \times [0, T)$, by Lemma 3.3, the function $x \mapsto u(x, \pi, t)$ is convex, so $u \leqslant x - K$ must be an interval of x, which clearly implies the expressions $\mathcal{C}$ and $\mathcal{R}$. The monotonicity of $X_1(\pi, t)$ and $X_2(\pi, t)$ is a simple consequence of (5.1) and (5.2). □

Similarly, for each fixed $(x, t) \in (0, \infty) \times [0, T)$, define

$$\Pi(x, t) := \max\{\pi \mid u(x, \pi, t) = (x - K)^+,\ (x, \pi, t) \in \mathcal{A}\}, \tag{5.8}$$

with $\Pi(x, t) = 0$ when the set on the right hand side is empty. Then by (5.1) and (5.2),

Lemma 5.2. *We have*

$$\mathcal{C} = \{(x, \pi, t) \in \mathcal{A} \mid \pi > \Pi(x, t)\},$$

$$\mathcal{R} = \{(x, \pi, t) \in \mathcal{A} \mid \pi \leqslant \Pi(x, t)\}.$$

Moreover $\Pi(x, t)$ is non-decreasing in t.

The target of this paper reduces to studying the properties of these redeeming boundaries $X_1(\pi, t)$, $X_2(\pi, t)$, and $\Pi(x, t)$.

Since $\gamma > r$, we only need to consider the following cases:

Case 0: $b \geqslant \gamma > r$

Case 1: $r \geqslant a$

Case 2: $\gamma > b > r$

Case 3: $\gamma > r > b$ and $a > r$

Case 4: $\gamma > r = b$

We start with the simplest Case 0.

5.1 Case 0: $b \geqslant \gamma > r$

If $b \geqslant \gamma > r$, a simple calculation shows for $x \geqslant K$

$$\begin{aligned} -\mathcal{L}(x-K) &= -\mathcal{L}x + \mathcal{L}K \\ &= (r - (\Delta\pi + b))x + (\gamma - r)K \\ &< (r-b)x + (\gamma - r)K \\ &\leqslant (r-b)K + (\gamma - r)K \\ &= (\gamma - b)K \\ &\leqslant 0, \end{aligned} \tag{5.9}$$

then we see that $\mathcal{R} = \emptyset$ and $\mathcal{C} = \mathcal{A}$.

Theorem 5.3. *Suppose $b \geqslant \gamma > r$, then the optimal redeeming time for the problem (2.7) is*

$$\tau^* = T.$$

Economically speaking, it is never optimal to early redeeming the stock because the loan rate is very low.

5.2 Case 1: $r \geqslant a$

Recalling our definition of the continuation region (2.9) and the redeeming region (2.10), we see from (5.3) that the redeeming boundary X_2 disappears if $r \geqslant a$.

Theorem 5.4. *If $r \geqslant a$, then*

$$\begin{aligned} \mathcal{C} &= \{(x, \pi, t) \in \mathcal{A} \mid x < X_1(\pi, t)\}, \\ \mathcal{R} &= \{(x, \pi, t) \in \mathcal{A} \mid x \geqslant X_1(\pi, t)\}. \end{aligned}$$

Economically speaking, one should redeem the stock if the sock price is high enough when the discounting rate is too high compared to the return rate of the stock.

Now we can summarize the result for the problem (4.1).

Theorem 5.5. *Suppose $r \geqslant a$, then the redeeming boundary can be expressed as $x = X_1(\pi, t)$, where $X_1(\pi, t)$ is non-decreasing in π, non-increasing in t, and $X_1(\pi, T) = K$ for any $0 \leqslant \pi \leqslant 1$. The optimal redeeming time for the problem (2.7) is given by*

$$\tau^* = \min\{\nu \in [t, T] \mid X_\nu \geqslant X_1(\pi_\nu, \nu)\}.$$

5.3 Case 2: $\gamma > b > r$

In this case the inequality in (5.3) may not hold.

Theorem 5.6. *Suppose $\gamma > b > r$, then*

$$X_1(\pi, t) > K, \quad X_2(\pi, t) < \tfrac{\gamma - r}{b - r} K. \tag{5.10}$$

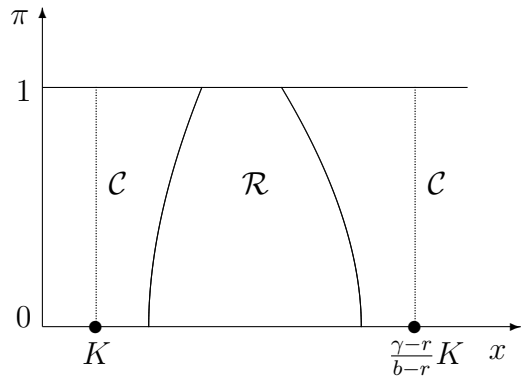


Figure 6: In the plan t-section, $\gamma > b > r$.

Proof. Recall (5.9),

$$-\mathcal{L}(x - K) = (r - (\Delta\pi + b))x + (\gamma - r)K, \tag{5.11}$$

and note $r - (\Delta\pi + b) < r - b < 0$, so if $-\mathcal{L}(x - K) \geqslant 0$, then

$$x \leqslant \tfrac{\gamma - r}{\Delta\pi + b - r} K < \tfrac{\gamma - r}{b - r} K.$$

Thus (5.10) follows from this and (2.11). □

Remark 5.1. *If $a > \gamma > b$, then $0 < \frac{\gamma - b}{\Delta} < 1$. Suppose $\pi \geqslant \frac{\gamma - b}{\Delta}$, then $\Delta\pi + b \geqslant \gamma$, hence, by (5.11), for $x > K$,*

$$-\mathcal{L}(x - K) = (r - (\Delta\pi + b))x + (\gamma - r)K \leqslant (r - \gamma)x + (\gamma - r)K < 0.$$

Recalling (2.11), we see that $\left\{(x, \pi, t) \in \mathcal{A} \mid \frac{\gamma - b}{\Delta} \leqslant \pi < 1\right\} \subseteq \mathcal{C}$ (see Figure 7).

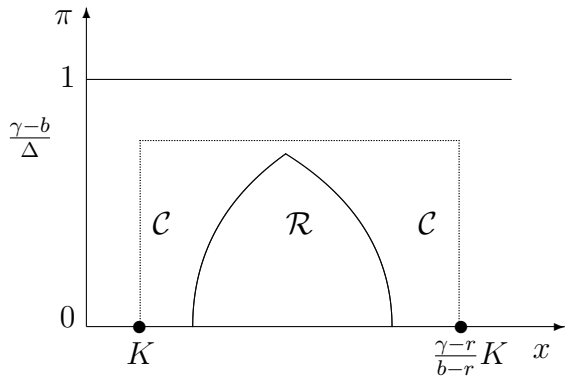


Figure 7: In the plan t-section, $a > \gamma > b > r$.

5.4 Case 3: $\gamma > r > b$ and $a > r$

We consider two subregions separately.

- $\frac{\gamma-b}{\Delta} \leqslant \pi < 1$: By Remark 5.1, if $a > \gamma > r > b$, then

$$\left\{(x,\pi,t) \in \mathcal{A} \mid \tfrac{\gamma-b}{\Delta} \leqslant \pi < 1\right\} \subseteq \mathcal{C}$$

  and hence no redeeming boundary in this region. The above relation holds when $\gamma \geqslant a > r > b$ as the left set becomes the empty set.

- $0 < \pi < \frac{\gamma-b}{\Delta}$: In this region if $\frac{(\gamma-r)K}{\Delta x} + \frac{r-b}{\Delta} < \pi < \frac{\gamma-b}{\Delta}$, we have $(\Delta\pi + b - r)x > (\gamma - r)K$, so by (5.11),

$$-\mathcal{L}(x-K) = (r - (\Delta\pi + b))x + (\gamma - r)K < 0,$$

  hence

$$\left\{(x,\pi,t) \in \mathcal{A} \mid \tfrac{(\gamma-r)K}{\Delta x} + \tfrac{r-b}{\Delta} < \pi < \tfrac{\gamma-b}{\Delta}\right\} \subseteq \mathcal{C}.$$

Recalling Lemma 5.2, we summarize the result obtained thus far in

Theorem 5.7. *If $\gamma > r > b$ and $a > r$, then*

$$\mathcal{C} = \{(x,\pi,t) \in \mathcal{A} \mid \pi > \Pi(x,t)\},$$
$$\mathcal{R} = \{(x,\pi,t) \in \mathcal{A} \mid \pi \leqslant \Pi(x,t)\},$$

where the redeeming boundary $\Pi(x,t)$ is first non-decreasing and then decreasing in x. The non-decreasing part is $x = X_1(\pi,t)$, and the non-increasing part is $x = X_2(\pi,t)$. Moreover, it is universally upper bounded by

$$\Pi(x,t) < \tfrac{\gamma-b}{\Delta}$$

when $a > \gamma$.

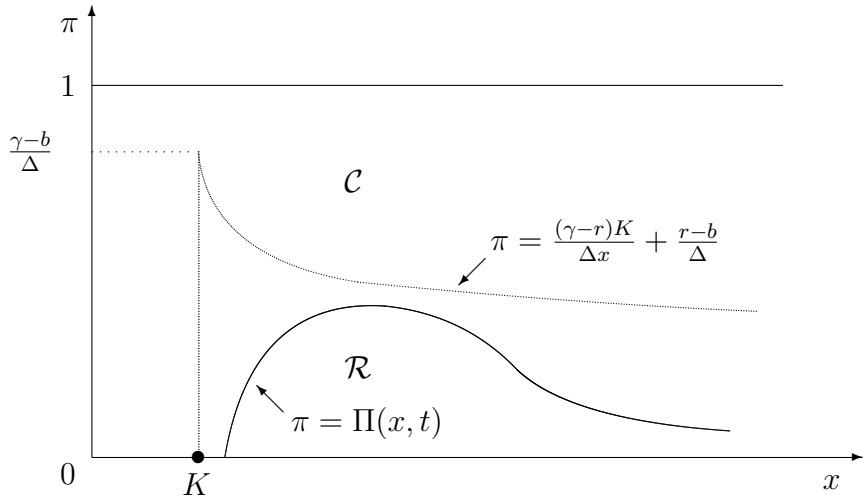


Figure 8: In the plan t-section, $a > \gamma > r > b$.

Economically speaking, one should not redeem the stock if excess return rate of the stock is very likely to be a (which is higher than the loan rate).

5.5 Case 4: $\gamma > r = b$

In this case, we have

Theorem 5.8. *If* $\gamma > r = b$, *then*

$$\mathcal{C} = \{(x, \pi, t) \in \mathcal{A} \mid \pi > \Pi(x,t)\},$$
$$\mathcal{R} = \{(x, \pi, t) \in \mathcal{A} \mid \pi \leqslant \Pi(x,t)\},$$

where the redeeming boundary $\Pi(x,t)$ *is first non-decreasing and then non-increasing in* x *with*

$$\lim_{x \to +\infty} \Pi(x,t) = 0.$$

And $\Pi(x,t) < \frac{\gamma-r}{\Delta}$ *when* $a > \gamma$.

Proof. We first prove that $\Pi(x,t)$ is positive for all sufficiently large x. Suppose not, because $\Pi(x,t_0)$ is non-increasing for large x, then there is a point $(x_0, 0, t_0)$ such that $\Pi(x,t_0) = 0$ for all $x > x_0$. From Proposition 4.3 and the definition of Π (5.8) we see that

$$u(x, 0, t_0) = x - K, \quad x > x_0;$$
$$u(x, \pi, t_0) > x - K, \quad x > x_0,\ \pi > 0.$$

By $u_t \leqslant 0$, we have

$$u(x,0,t) = x - K, \quad x \geqslant x_0,\ t \geqslant t_0; \tag{5.12}$$

$$u(x,\pi,t) > x - K, \quad x \geqslant x_0,\ \pi > 0,\ t \leqslant t_0. \tag{5.13}$$

Hence

$$u_t + \tfrac{1}{2}x^2u_{xx} + \tfrac{1}{2}\Delta^2\pi^2(1-\pi)^2u_{\pi\pi} + \Delta\pi(1-\pi)xu_{x\pi} + (\Delta\pi + b - \gamma)xu_x + (\gamma - r)u = 0, \quad x \geqslant x_0,\ \pi > 0,\ t \leqslant t_0.$$

Letting π go to 0,

$$u_t + \tfrac{1}{2}x^2u_{xx} + (b-\gamma)xu_x + (\gamma - r)u = 0, \quad x \geqslant x_0,\ \pi = 0,\ t \leqslant t_0. \tag{5.14}$$

By (5.12), we have for $x \geqslant x_0$, $\pi = 0$, $t = t_0$,

$$u_t(x,0,t_0) \leqslant 0, \quad u_{xx}(x,0,t_0) = 0, \quad u_x(x,0,t_0) = 1, \quad u(x,0,t_0) = x - K.$$

Substituting them into the equation (5.14) with $t = t_0$,

$$u_t(x,0,t_0) + (b-\gamma)x + (\gamma - r)(x - K) = 0.$$

Applying $r = b$, we obtain $u_t(x,0,t_0) = (\gamma - r)K > 0$, a contradiction.

The non-decreasing part of $\pi = \Pi(x,t)$ corresponds to $x = X_1(\pi,t)$ in (5.4), and the non-increasing part corresponds to $x = X_2(\pi,t)$ in (5.5).

For $\pi > \frac{(\gamma-r)K}{\Delta x}$, we have by (5.11),

$$-\mathcal{L}(x-K) = (r - (\Delta\pi + b))x + (\gamma - r)K = -\Delta\pi x + (\gamma - r)K < 0,$$

hence by (2.11)

$$\left\{(x,\pi,t) \in \mathcal{A} \mid \pi > \tfrac{(\gamma-r)K}{\Delta x}\right\} \subseteq \mathcal{C}.$$

This clearly implies $\lim_{x\to+\infty}\Pi(x,t) = 0$ and $\Pi(x,t) < \frac{\gamma - r}{\Delta}$ when $a > \gamma$ as $\Pi(x,t) > K$.□

See Figures 9, 10 for illustrations of our result when $a \leqslant \gamma$ and $a > \gamma$.

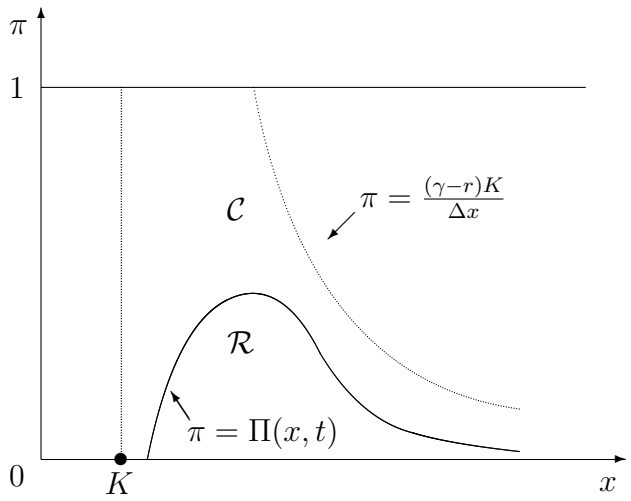


Figure 9: In the plan t-section, $\gamma \geqslant a > r = b$.

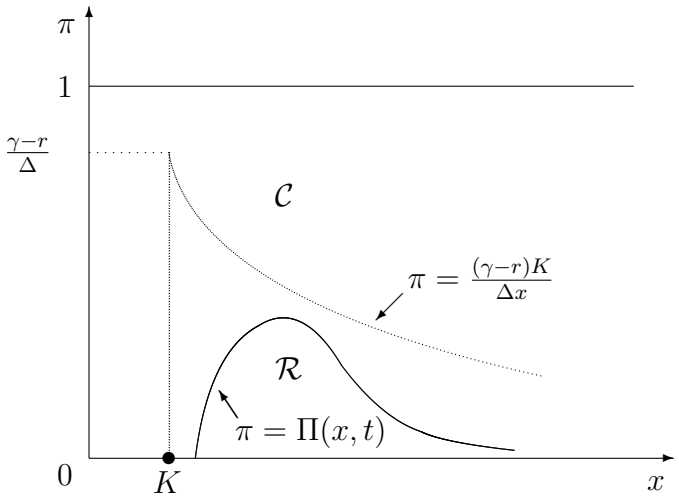


Figure 10: In the plan t-section, $a > \gamma > r = b$.

6 Concluding remarks

Due to space limit, this paper has only studied the optimal redeeming problem of the standard stock loan. Clearly, one can use our method to study stock loans with other ways of dividend distribution. Such problems, however, will involve higher dimensional processes so that more involved analysis is required. We hope to investigate them in future works.

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