ICS 2011
12th INFORMS Computing Society Conference

© 2011 INFORMS | ISBN 978-0-9843378-1-1
DOI 10.1287/ics.2011.0037

Assignment Preferences and Combinatorial Auctions

Robert W. Day
Operations and Information Management, School of Business, University of Connecticut, Storrs, Connecticut 06269, bob.day@business.uconn.edu

S. Raghavan
The Robert H. Smith School of Business and Institute for Systems Research, University of Maryland, College Park, Maryland 20742, raghavan@umd.edu

Abstract This paper considers auctions for several distinct items in which each bidder's valuation function is determined by an optimal assignment of goods among several agents, each with an independent valuation for each good. Given this preference structure, we demonstrate how to compute the set of lowest Walrasian equilibrium prices, generalizing the work of Demange et al. (G. Demange, D. Gale, and M. Sotomayor. Multi-item auctions. *Journal of Political Economy* 94(4):863–872, 1986), which considered the special case in which bidders are only interested in receiving a single item.

In the more general combinatorial auction setting, where bidders may have arbitrary valuation functions, we propose that the resulting "bid table" bidding language provides a useful communication format for use in a dynamic demand-revelation phase of a multistage hybrid auction. This new format for demand revelation results in unique linear item prices which can be computed in polynomial time, and with bidder input growing quadratically in the number of items. Relative to the simultaneous ascending auction used in combinatorial auctions by the FCC, this can be accomplished without the exposure to receiving substitute goods at additive prices, and without the ability for competitors to signal among themselves.

Keywords combinatorial auctions; assignment problem; gross substitutes; Walrasian equilibrium; duality; bidder-optimal prices; bidding languages

1. Introduction

In a combinatorial auction, an auctioneer offers for sale a set of goods, and after some prespecified bidding process takes place, bidders are awarded subsets of this set. These subsets are often referred to as packages, bundles of goods, or combinations of goods, giving rise to alternative terms for combinatorial auctions, such as package auctions, combinational auctions, etc. Combinatorial auctions have been used extensively in practice in recent years and have a well developed stream of academic literature recognizing the various economic and computational challenges of implementing such auctions. Though we refer the reader to a survey by Anandalingam et al. [1] or the edited volume by Cramton et al. [9] for extensive background and motivation, we note that the primary motivation for combinatorial auctions is the existence of bidders with nonadditive preferences for bundles of goods. That is, in some cases bidders think of items as substitutes, so that their willingness to pay for several items is less than the sum of the their willingness to pay for each, or in other cases as complements, in which case the value for the collection of items is more than the sum of the values for the individual items.

The central computational problem in a combinatorial auction is to find a value-maximizing collection of winning bids, often referred to as the efficient allocation problem or the winner-determination problem. In the most general case, this optimization problem (performed by the auctioneer) is equivalent to the set-packing problem and thus NP-hard; see Lehmann et al. [19]. Though this computational difficulty can be reasonably ignored in certain applications that have a relatively small number of items for sale, it has motivated authors to investigate tractable special cases of the winner-determination problem; see Müller [20] for several such special cases.

This paper examines the properties of one such special class of winner-determination problems, in which each bidder's preferences are determined by an optimal assignment of goods among several agents, each with an independent valuation for each good. Given this preference structure, we demonstrate how to compute the set of lowest Walrasian equilibrium prices, generalizing the work of Demange et al. [12], which considered the special case in which bidders are only interested in receiving a single item. The strengths and limitations of the "bid table" language corresponding to this "assignment preference" structure are shown via illustrative examples and a classification result. The latter states that the set of preferences expressible in bid tables are properly contained in the set of preferences satisfying the "gross substitutes property," allowing the application of a theoretical result of Ausubel and Milgrom [3]. This result elucidates a strength of a bid table auction, that a Vickrey-Clark-Groves (VCG) pricing mechanism may be used with no possibility of disruption from false-name bidding or joint deviation by losing bidders, properties not satisfied for the VCG mechanism in general.

Further, we provide a context in which bid tables can be useful in the general combinatorial auction setting to reveal linear price signals (i.e., prices for individual items which can be added to get a price for the whole bundle) and protecting bidders from certain "exposure problems," for example, the possibility of receiving multiple substitute goods. In Day and Raghavan [11], we introduced a multistage combinatorial auction using a dynamic bid table auction as an initial demand revelation phase. Bid tables allow us to utilize the intuitive linear prices as long as it makes sense to do so, until any further bidding requires a departure from linear prices in order to treat complementary preferences. Relative to the simultaneous ascending auction (SAA), this can be accomplished without the exposure to receiving substitute goods at additive prices, and without the ability for competitors to signal among themselves.

The remainder of the paper is organized as follows. In §2, we provide relevant background on combinatorial auctions, linear prices, and two special classes of combinatorial auctions studied in the literature for which linear price equilibria exist. In §3, we introduce assignment preferences and bid tables, suggesting some of their potential uses, with the gross substitutes characterization following in §4. In §§5 and 6 we discuss VCG and dynamic bid table auction implementations, respectively, with conclusions provided in §7.

2. Background

2.1. Combinatorial Auctions

A general combinatorial auction has N types of items for sale, with the set of item types denoted $I = \{1, 2, \dots i, \dots N\}$, and a set of M bidders referred to as $J = \{1, 2, \dots j, \dots M\}$. For the sake of generality we allow for several identical copies of each item to be present in the auction, with a single seller in the auction supplying the quantity $\sup_i$ of each item i. For any feasible bundle or package of items $\mathbf{x} \in \mathcal{S} = \{\mathbf{x} \in \mathbb{Z}^N \mid 0 \le x_i \le \sup_i, \forall\, i \in I\}$, each bidder j perceives some value $v_j(\mathbf{x}) \ge 0$ equal to her utility of receiving this bundle at zero cost, with the standard assumption that $v_j(\mathbf{0}) = 0$, $\forall\, j$. Further, as is common in the combinatorial auction literature, we assume free-disposal (i.e., $v_j(\mathbf{x} + \mathbf{e}^i) - v_j(\mathbf{x}) \ge 0$, $\forall\, i, \mathbf{x}$, where $\mathbf{e}^i$ is the unit vector with a 1 in the ith position) and define a bidder's net utility as

$u_j(\mathbf{x}, p_j) = v_j(\mathbf{x}) - p_j$, where p_j is the payment made by bidder j. This latter (standard) assumption on the functional form of a bidder's utility function is usually referred to as *quasilinear* net utility, since utility is defined to be linear over payments but potentially nonlinear over items. Also, since the bid that a bidder submits on package $\mathbf{x}$ might differ from her true valuation, the monetary bid on $\mathbf{x}$ will be denoted $b_j(\mathbf{x})$, indicating that the bidder has offered to pay up to this amount to receive the bundle $\mathbf{x}$.

In an *efficient* combinatorial auction, an allocation of bundles to bidders is chosen to maximize the sum of the values of the corresponding bids accepted by the auctioneer, while not allocating more items than are available. Also, to maintain consistency with the specified bids and allow for a fully expressive communication of preferences, it is a standard to assume an XOR bidding language (see Nisan [22]) for the general combinatorial auction problem, meaning that at most one bid can be accepted from each bidder. (This forbids the possibility, for example, that the auctioneer accepts bids from bidder j for both $\mathbf{x}^1$ and $\mathbf{x}^2$ when the bidder has specified bid amounts such that $b_j(\mathbf{x}^1) + b_j(\mathbf{x}^2) > b_j(\mathbf{x}^1 + \mathbf{x}^2)$.) A typical integer programming formulation of the *efficient allocation problem* or *winner-determination problem* is thus stated as follows:

$$\max \sum_{x \in \mathcal{S}} \sum_{j \in J} b_j(\mathbf{x}) y_j(\mathbf{x}) \tag{WDP}$$

$$\text{subject to} \sum_{x \in \mathcal{S}} \sum_{j \in J} x_i y_j(\mathbf{x}) \leq \text{sup}_i, \quad \forall i \in I, \tag{1}$$

$$\sum_{x \in \mathcal{S}} y_j(\mathbf{x}) \leq 1, \quad \forall j \in J, \tag{2}$$

$$\text{where } y_j(\mathbf{x}) = \begin{cases} 1 & \text{if package } \mathbf{x} \text{ is awarded to bidder } j, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where constraints (1) ensure that the auctioneer does not sell more copies of an item than are available, while constraints (2) maintain the XOR language, ensuring that at most one bid is accepted per bidder. This formulation assumes a fixed set of bids (i.e., the only variables are the $y_j(\mathbf{x})$ variables), and in a *sealed-bid* implementation of a combinatorial auction, the bidders may simply submit the bids on each package all at once and the auctioneer solves this problem.

2.2. Linear Prices

In contrast to sealed-bid auctions, in many applications it is desirable to have *price discovery* or *demand revelation*, in which bidders iteratively submit bids, receive feedback (typically in the form of prices, and usually a linear price for each item), and revise their bids accordingly to match new prices or restrictions, etc. In this case we also consider a price vector $\mathbf{p}$ where each component p_i denotes a price for item i, and the resulting analogous linear-price utility function for bidder j is expressed as $u_j(\mathbf{x}, \mathbf{p}) = v_j(\mathbf{x}) - \sum_{i \in I} p_i x_i$. Also in the environment of linear prices, we say that a bundle of items $\mathbf{x}$ is *demanded by j at a price* $\mathbf{p}$ if $\mathbf{x}$ maximizes net utility for bidder j at $\mathbf{p}$. By denoting the set of all bundles demanded by j at price $\mathbf{p}$ as $D_j(\mathbf{p})$, we may write this mathematically as

$$D_j(\mathbf{p}) = \arg\max_{x \in \mathcal{S}} u_j(\mathbf{x}, \mathbf{p}).$$

A bidder may demand several bundles at $\mathbf{p}$ if each maximizes net utility, and demands $\mathbf{0} = [0, 0, \ldots 0]$ when prices are too high.

In order for linear prices to be a useful indicator of demand in the auction, it is typical to want prices that are *accurate* and *separate* winning bundles from losing bundles. That is,

if the auction were to close immediately, we want the current prices to reflect the actual payments made by winners (accuracy) and for the current price of any bundle to be higher than what those not winning it are willing to pay (separation). Accuracy simply requires that each bidder's payment and the allocated bundle, denoted $\mathbf{x}^j$, satisfy $p_j = \sum_{i \in I} p_i x_i^j$. Separation requires that $\mathbf{x}^j \in D_j(\mathbf{p})$ for all bidders j. In this environment, an allocation with accurate and separating prices form what is known as a Walrasian equilibrium.

Definition 1. An allocation of bundles to bidders $(\mathbf{x}^1, \mathbf{x}^2, \ldots \mathbf{x}^j \ldots \mathbf{x}^M)$ and price vector $\mathbf{p}$ constitute a *Walrasian equilibrium* if and only if for every bidder j, $\mathbf{x}^j \in D_j(\mathbf{p})$.

Unfortunately, Walrasian equilibrium prices may not exist in a combinatorial auction when items are complements, as demonstrated by the following example, where we assume one copy of each item and use set notation (rather than vector form) to denote a bundle. That is, for the sake of brevity, we use the set notation $\{B, C\}$ in place of the vector $[0, 1, 1]$, etc.

Example 1. In a four-bidder, three-item auction let the bids on items A, B, and C be as follows:

$$b_1\{A, B, C\} = 6, \qquad b_2\{A, B\} = 5,$$
$$b_3\{A, C\} = 5, \qquad b_4\{B, C\} = 5.$$

Clearly, the efficient solution is to award all three items to bidder 1, but what prices can be assigned to the individual items that separate the winner from the losers? In order for the losing bidders to be satisfied, the sum of the prices of the items in a bundle should exceed any losing bid on the bundle:

$$p_A + p_B \geq 5,$$
$$p_A + p_C \geq 5,$$
$$p_B + p_C \geq 5.$$

But these inequalities imply that $p_A + p_B + p_C \geq 7.5$, a total payment that is too high for bidder 1, who will pay at most 6.

This example illustrates the well known failure of linear prices, that there may exist no linear prices to support a Walrasian equilibrium. At the conclusion of a combinatorial auction allowing for the most general expression of preferences, separating prices can only be expressed in terms of bundle-payments made by the winners and cannot be decomposed into meaningful individual item prices.

2.3. Special Cases

Walrasian equilibrium prices are guaranteed to exist in certain special cases of a combinatorial auction, however, and here we note the two most relevant such cases as previously studied in the literature. First, when each bidder is interested in winning at most one item, a condition referred to as *unit-demand* preferences, the resulting multiunit auction has been called "the assignment game" and studied extensively in the context of matching markets. In this special case we have $v_j(\mathbf{x}) = \max_{e^i \leq x} v_j(\mathbf{e}^i)$ for all j and $\mathbf{x}$, and a bidder's preferences can be compactly represented by the list of values for each individual item. The winner-determination problem can then be formulated as an assignment problem (also know as maximum weight bipartite matching), which is known to have a totally unimodular constraint matrix, and thus the linear programming (LP) relaxation solves the integer winner-determination problem. The relevant results from this stream of literature tell us that any set of prices from the dual to this LP relaxation forms a Walrasian equilibrium, and that the well-known Hungarian algorithm finds the unique "bidder-optimal" or "minimal" Walrasian equilibrium prices for the case of unit-demand bidders. For further background on

this material see Demange et al. [12], Roth and Sotomayor [23, Chapter 8], and Nemhauser and Wolsey [21, pp. 540–544].

This paper studies many of the same economic properties as Roth and Sotomayor [23] and Demange et al. [12], but we move from the unit-demand model to a more general case in which bidders are represented by *more than one* unit-demand "agent" (i.e., each agent is just a single unit-demand bidder), and experience the sum of the utilities each of their own agents gets for being matched to an item. The Hungarian algorithm will not provide minimal Walrasian equilibrium prices in this larger context, but we provide an algorithm to compute such minimal prices in §6.

Bikhchandani and Ostroy [6] illustrate a more general set of special cases: when the LP relaxation of problem WDP solves to integral optimality, the LP dual to this relaxation provides Walrasian equilibrium prices (based on bids rather than true preferences). This dual can be written as follows:

$$\min \ \sum_{i\in I} \sup_i p_i + \sum_{j\in J} s_j \tag{WDP-D}$$

$$\text{subject to } \sum_{i\in I} x_i p_i + s_j \geq b_j(\mathbf{x}), \quad \forall\, j \in J,\ \forall\, \mathbf{x} \in \mathcal{S}, \tag{4}$$

$$p_i \geq 0, \quad \forall\, i \in I, \tag{5}$$

$$s_j \geq 0, \quad \forall\, j \in J. \tag{6}$$

Based on complementary slackness conditions, p_i variables can indeed be interpreted as linear item prices, while s_j variables indicate bidder j's surplus, or observed net utility, and the following statements are true: When bidder j is awarded $\mathbf{x}^j$, her surplus is her bid minus her payments. When an item goes unsold its price must equal zero. When a bidder is awarded no bundle, her surplus must equal zero.

From constraints (1), we also have that for any bundle a bidder j does not receive, the prices are such that j's bid on this bundle minus its price (i.e., bidder j's perceived surplus for this bundle) is not greater than the surplus for the bundle actually awarded. This verifies that each awarded bundle belongs to the bidder's demand set (based on submitted bids) thus verifying the Walrasian equilibrium property.

This paper also investigates a special class of winner-determination problems for which the LP relaxation solves to integral optimality. Unlike the case described by Bikhchandani and Ostroy, however, we use a formulation of winner determination that is quadratic (rather than exponential) in the number of items, and find that not all of the same properties hold. In particular, we will see that although the LP dual of the winner-determination problem for assignment preferences does contain *only* Walrasian equilibrium prices, it does not contain *all* such prices, and in particular may omit the most important, bidder-optimal prices.

3. Assignment Preferences and Bid Tables

We now describe a new compact method for a bidder to write down bid information as an alternative to assigning a price to every bundle explicitly (since this expression grows exponentially in the number of items being auctioned). The approach for preference elicitation explored here makes use of the concept of a *price-vector agent*. Each agent can be thought of as a fictional entity representing some portion of the preferences of a particular bidder. A price-vector agent is assigned a vector of prices (a monetary amount for each of the items in the auction) and "participates" in the auction based on these prices. An agent receiving a particular item pays at most the price vector component associated with that item. Throughout, we assume that each price-vector agent is a unit-demand agent, receiving at most one item. This clarifies the role of the price-vector agent in economic terms: each agent treats all items as perfect substitutes, meaning that only one of them provides any

value to that agent. Similarly, bid tables allow for expression of partial substitutability, in which the incremental value of additional items diminishes as more items are received.

Suppose that, each bidder j has a set of A_j agents that she potentially wishes to satisfy, denoted by the set $K_j = \{1, 2, \ldots k, \ldots A_j\}$. Let v_{ijk} denote the utility perceived by bidder j when agent k receives item i. The preference assumption on $v_j(\mathbf{x})$ which we refer to as *assignment preferences* is specified as follows:

$$
\begin{aligned}
v_j(\mathbf{x}) = \max \ & \sum_{i \in I} \sum_{k \in K_j} v_{ijk} y_{ijk} && \text{(AP)} \\
\text{subject to } & \sum_{i:\, x_i \geq 1} y_{ijk} \leq 1, \quad \forall\, k \in K_j, \\
& \sum_{k \in K_j} y_{ijk} \leq x_i, \quad \forall\, i \in I, \\
& \text{where } y_{ijk} = \begin{cases} 1 & \text{if item } i \text{ is assigned to bidder } j\text{'s } k\text{th agent,} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

In short, a bidder's value for a bundle of items is a maximal value assignment of those items to her agents. As is well known (see Nemhauser and Wolsey [21, p. 544]), the constraint matrix of this integer program is totally unimodular, indicating that the problem can be solved to integral optimality by its LP relaxation. This implies that a bidder could rapidly determine her value for any set of items with an LP solver or faster combinatorial algorithm designed specifically to solve assignment problems, even for a large value of N.

We call the collected set of (column) price-vectors a *bid table*. The result is an easy-to-read method of compactly annotating certain forms of preferences for substitute goods. To interpret a bid table, one need only keep in mind that at most one bid entry may be accepted from each row, and at most one from each column.

As an example, with the bid table of Figure 1 we may determine a bidder's value for any bundle of items a, b, c, or d if assigned to any of three possible agents. Considering the values in this bid table, we notice that a bidder's value for any single item is simply the maximum value for that row in the bid table; if a bidder is awarded only one item, the agent that experiences the most utility from this item will be accommodated. Thus item a by itself is worth 2 units to the bidder, item b by itself is worth 5 units, item c by itself is worth 6 units, and item d by itself is worth 3 units. The value for some collection of items is not, however, necessarily equal to the sum of her values for individual items, as there may be conflicts when the row maximums occur in the same column. For example, for bidder j with this bid table $b_j\{b\} = 5$, $b_j\{c\} = 6$ but $b_j\{b, c\} = 9 \neq 5 + 6$. We find $b_j\{a, b, c, d\} = 11$ with an optimal assignment of a to the third agent, b to the second agent, and c to the first agent. We notice that to achieve this amount we do not assign item c to the second agent, despite this being the overall highest value in the bid table, and also that we choose to assign item a and not item d, despite the fact that $b_j\{a\} < b_j\{d\}$.

FIGURE 1. A bid table.

	Agent 1	Agent 2	Agent 3
Item a	0	0	2
Item b	2	5	3
Item c	4	6	4
Item d	0	3	0

Notes. Each entry is a dollar amount offered for the good in the row. At most one entry can be accepted from each row, and at most one from each column.

FIGURE 2. This bidder therefore only offers a positive amount for one item from the set $\{A_1, A_2, A_3\}$, and offers incrementally less for items $\{B_1, B_2, B_3\}$ as more of them are taken together.

	Agent 1	Agent 2	Agent 3	Agent 4
Pure substitute A_1	15	0	0	0
Pure substitute A_2	16	0	0	0
Pure substitute A_3	17	0	0	0
Partial substitute B_1	0	23	18	12
Partial substitute B_2	0	20	16	10
Partial substitute B_3	0	19	15	9

These observations show that assignment preference valuation functions can have some nonintuitive properties and are not contained in the class of additive valuation functions (in which the value of some set of items always equals the sum of the values for the individual items). We notice that any additive valuation function can be modeled by a bid table by taking constant rows and at least as many agents as there are items. Thus additive valuation functions are properly contained in the class of assignment preference valuation functions.

Figure 2 shows an example of a bid table and demonstrates how a single agent can be used to name prices for pure substitutes, or how a collection of agents can combine to yield decreasing offers on partially substitutable items. We see therefore that in the example of Figure 2 that items A_1, A_2, and A_3 are indeed pure substitutes, because at most one can be purchased at a positive price. Similarly, if the bidder of this example receives item B_1 priced at 23, she cannot also receive (for example) item B_2 at a price of 20. If item B_2 is assigned to the bidder, the revenue maximizing auctioneer would be forced to accept a lower price from another agent of that bidder. In this case the auctioneer would collect 16 for B_2 rather than 20, verifying the partial substitutability; B_2 is worth less if taken with B_1.

The role of each column or price vector agent varies from application to application, but in some scenarios the agents may have a very natural interpretation. Motivated by the proposed use of a combinatorial auction for the allocation of airport landing slots (see Ball et al. [5]), it may be useful to think of each agent as representing a potential flight, for example, and each item as a slot made available to an airline at an airport for landing a single plane. In this case, unit-demand agents representing each airplane should seem natural since each is interested in consuming at most one slot to land the plane.

As another example, consider an auto-trader bidding for used cars who wants to obtain at most one SUV and one sedan, but has varying preferences over colors. Suppose he likes black cars more than white ones, and white ones more than red ones. This auto-trader's preferences may look like those given in Figure 3. By using separate agents for his SUV and sedan preferences he can assure that he doesn't get more than one of either. This example also motivates a real world interpretation of assignment preferences, in which a real auto-trader might send out two different agents to an actual auction, each with different interests in mind to achieve a combined goal.

FIGURE 3. A bid table for used cars, entries in $1,000s.

	SUV agent	Sedan agent
Black SUV	22	0
White SUV	20	0
Red SUV	17	0
Black sedan	0	13
White sedan	0	10
Red sedan	0	9

Note. This bidder is interested in at most one SUV and at most one sedan, but has varying bids based on color.

4. Assignment Preferences and the Gross Substitutes Property

One may next ask how assignment preferences relate to the larger class of valuation functions which satisfy the gross substitutes property (as discussed in Kelso and Crawford [17] for example).

Definition 2. The *gross substitutes property* holds if and only if the following condition holds for every bidder j: For any price vectors $\mathbf{p}' \geq \mathbf{p}$ with $\mathbf{p}' \neq \mathbf{p}$, and any $\mathbf{x} \in D_j(\mathbf{p})$, there exists an alternative $\mathbf{x}' \in D_j(\mathbf{p}')$ with $x'_i \geq x_i$ for all i with $p'_i = p_i$.

Or roughly, if the prices rise on some of the items in a demanded set, then there is at least one demanded bundle at the new prices still containing all previously demanded items for which the price did not increase.

The gross substitutes property is sufficient to guarantee the convergence of several ascending price multi-item auction formats to a Walrasian equilibrium (e.g., those of Ausubel and Milgrom [3], Gul and Stachetti [15]) and has other beneficial properties which will be discussed below. Though it is a common assumption due to its attractive theoretical properties, it is uncommon for theorists to describe applications which give rise to this property, to explain under what conditions it is a safe assumption to make, or to suggest how to enforce this restriction among bidders if it is necessary for the convergence of an auction to a desirable outcome. In this paper, we start instead from the assumption of assignment preferences and show that the gross substitutes property follows as a result.

Theorem 1. *A valuation function determined by assignment preferences displays the gross substitutes property.*

Proof. Suppose we begin at price vector $\mathbf{p}^1$ with a particular demanded bundle $\mathbf{x}^* \in D_j(\mathbf{p}^1)$ and consider price vector $\mathbf{p}^2 \geq \mathbf{p}^1$, with $\mathbf{p}^2 \neq \mathbf{p}^1$ and a particular demanded bundle $\mathbf{x}' \in D_j(\mathbf{p}^2)$. Under assignment preferences a demanded set is always supported by an assignment of items to agents which maximizes total value net of prices (sometimes more than one supporting assignment may exist). Begin by selecting particular assignments (sets of arcs to item agent pairs) A^* and A' supporting $\mathbf{x}^*$ and $\mathbf{x}'$, respectively. We will say that arc $i \to k$ is in A if assignment A has item i assigned to agent k.

Now, given an element $\bar{\imath}$ with $x^*_{\bar{\imath}} \geq 1$ with $p^1_{\bar{\imath}} = p^2_{\bar{\imath}}$ and $x'_{\bar{\imath}} < x^*_{\bar{\imath}}$, we show how to find a demanded bundle $\mathbf{x}^{**} \in D_j(\mathbf{p}^2)$ with $x^{**}_{\bar{\imath}} \geq x^*_{\bar{\imath}}$ and corresponding assignment A^{**}. We do this by showing how to construct two sets IN and OUT such that $A^{**} = (A' \setminus OUT) \cup IN$.

Algorithm 1 (Constructing a $\mathbf{x}^{**} \in D_j(\mathbf{p}^2)$ containing at least as much of $\bar{\imath}$).

Step 0. Set $InItem = \bar{\imath}$, $OutItem = IN = OUT = \varnothing$.
Step 1. Let k be an agent assigned $InItem$ under A^*.
Step 2. Find an item (or $\varnothing$) $OutItem$ with $OutItem \to k$ in A'.
Step 3. Add arc $OutItem \to k$ (or $\varnothing$) to OUT. Add arc $InItem \to k$ to IN.
Step 4. If $OutItem \notin \mathbf{x}^*$ (or $OutItem = \varnothing$), terminate, else set $InItem = OutItem$ and goto Step 1.
Terminate: Set $A^{**} = (A' \setminus OUT) \cup IN$.

Claim 1. The bundle $\mathbf{x}^{**}$ implied by arc set A^{**} is in $D_j(\mathbf{p}^2)$. Suppose not. Then $\mathbf{x}' \in D_j(\mathbf{p}^2)$ provides $u_j(\mathbf{x}^{**}, \mathbf{p}^2) < u_j(\mathbf{x}', \mathbf{p}^2)$. Since by construction each member of OUT was in A', this yields $u_j(IN, \mathbf{p}^2) < u_j(OUT, \mathbf{p}^2)$, with the obvious interpretation of the utility for an arc set. That is

$$u_j(IN, \mathbf{p}^2) = \sum_{i \to k \in IN} v_{ijk} - \sum_{i \,|\, i \to k \in IN} p^2_i < u_j(OUT, \mathbf{p}^2) = \sum_{i \to k \in OUT} v_{ijk} - \sum_{i \,|\, i \to k \in OUT} p^2_i.$$

But because every item assigned under IN is assigned under OUT, except for $\bar{\imath}$ which does not experience a price change from $\mathbf{p}^1$ to $\mathbf{p}^2$, we have $u_j(IN, \mathbf{p}^1) < u_j(OUT, \mathbf{p}^1)$. But by

construction of these arc sets $(A^*\backslash IN) \cup OUT$ is a feasible assignment, and then we must have $u_j(A^*, \mathbf{p}^1) < u_j((A^*\backslash IN) \cup OUT, \mathbf{p}^1)$, contradicting our choice of $\mathbf{x}^* \in D_j(\mathbf{p}^1)$.

The validity of the claim provides a demanded set containing $\bar{\imath}$. Each execution of the algorithm finds a set in $\mathbf{x}^{**} \in D_j(\mathbf{p}^2)$ containing an element from $\mathbf{x}^*$ that was missing in $\mathbf{x}'$ and does not remove any elements from $\mathbf{x}^* \cap \mathbf{x}'$. By repeating this procedure (setting $\mathbf{x}' = \mathbf{x}^{**}$ each time) we arrive at a set in $D_j(\mathbf{p}^2)$ containing all the desired elements of $\mathbf{x}^*$. □

To illustrate the idea of the proof consider the following example in which the items in the auction are denoted by lowercase letters. Suppose at prices $\mathbf{p}^1$ the bidder in question demands the bundle $\{a, b \times 2, c, d, e, f\}$, (where $b \times 2$ indicate two copies of item b in the bundle) which is found to maximize utility with the assignment of items to agents $A^* = \{a \to 1,\ b \to 2,\ c \to 3,\ d \to 4,\ de \to 5,\ f \to 6,\ b \to 7\}$. Suppose next that prices rise on items a and d. The bidder then recomputes for an optimal bundle at these prices and finds the optimal assignment $A' = \{h \to 1,\ r \to 2,\ g \to 3,\ q \to 4,\ b \to 5,\ d \to 6,\ c \to 7\}$, but this assignment clearly does not validate the gross substitutes property; items e and f did not experience a price increase but they do not appear in the newly demanded bundle, while item b did not experience a price increase but had a reduction in demand from 2 to 1.

To apply the algorithm described in the proof, first consider finding a demanded set including two copies of item b. Putting A^* above A' and designating destination agents by column, we see below that agent 2 who was assigned b in A^* is assigned r in A'.

A^*	:	a	b	c	d	e	f	b
			$\downarrow$					
A'	:	h	r	g	q	b	d	c
—	—	—	—	—	—	—	—	—
A^{**}	:	h	b	g	q	b	d	c

The manipulation performed by the algorithm is in its simplest form here. The value of A^{**} at $\mathbf{p}^2$ must be at least as much as the value of A' at $\mathbf{p}^2$, or else it would be possible to switch r in for b in A^* and receive a higher value at $\mathbf{p}^1$, contradicting the optimality of A^* as a demanded bundle. This reasoning only holds because the price of b does not change in the movement from $\mathbf{p}^1$ to $\mathbf{p}^2$, and because r can be switched in freely as it is not in A^*.

When we try to find a demanded bundle at $\mathbf{p}^2$ containing e, it is not quite so easy; the agent assigned e under A^* is assigned b under A', and since b is already assigned under A^* a one-for-one switching argument fails. The algorithm rectifies this by tracing back a path until an item is found that was not allocated under A^*. For example, the following diagram helps us see that in moving from A^* to A' (which has now been replaced by A^{**} from the previous step) item e's agent is reassigned item b, whose agent in A^* is reassigned item c, whose agent in A^* is reassigned by item g which was unallocated in A^*. (Note that if we picked agent 2 for item b rather than agent 7, we simply add one more trivial step before finding the item g unallocated in A^*.) The same optimality argument can be used to show a demanded bundle at $\mathbf{p}^2$ including item e; $\{g \to 3, b \to 5, c \to 7\}$ can be replaced by $\{c \to 3, e \to 5, b \to 7\}$ or else the optimality of A^* is contradicted.

A^*	:	a	b	c	d	e	f	b
				$\downarrow$		$\downarrow$		$\downarrow$
A'	:	h	b	g	q	b	d	c
—	—	—	—	—	—	—	—	—
A^{**}	:	h	b	c	q	e	d	b

FIGURE 4. Preferences that do not fit in a bid table.

$$\begin{array}{ll} v\{a\} = 10, & v\{a,b\} = 20 \\ v\{b\} = 12, & v\{a,c\} = 19 \\ v\{c\} = 13, & v\{b,c\} = 17 \end{array}$$

a	10	8?
b	12	10?
c	13	

Note. The value $v\{a,b\} = 20$ requires either an 8 or 10 to be inserted as shown, but either attempt results in the over-valuation of another bundle.

Arguing similarly allows us to settle on a final set A^{**}, demanding all items for which prices did not increase with at least the same levels as at price $\mathbf{p}^1$. The final step is displayed in the following diagram.

$$\begin{array}{lcccccccc} A^* & : & a & b & c & d & e & f & b \\ & & & & & \downarrow & & \downarrow & \\ A' & : & h & b & c & q & e & d & b \\ — & — & — & — & — & — & — & — & — \\ A^{**} & : & h & b & c & d & e & f & b \end{array}$$

Lehmann et al. [18] independently showed that the OR-of-XOR of singletons language satisfies the gross substitutes property. This logically constructed language turns out to be equivalent to assignment preferences, thus providing an indirect way of stating our result. Further, concurrently to the Ph.D. dissertation of Day [10] which includes portions of this paper, Hatfield and Milgrom [16] made a similar connection between assignment preferences and gross substitutes.

In contrast to these alternative proof techniques, our proof of the gross substitutes property is constructive and thus useful in its own right algorithmically. Experience shows that us that using an LP solver to determine demanded bundles under assignment preferences may give any demanded bundle after a price increase and will often not provide a bundle that verifies the validity of the theorem. For any demanded bundle $\mathbf{x}^*$ before a price increase and a given demanded bundle $\mathbf{x}'$ after the price increase, the algorithm described in the proof constructs a new demanded bundle $\mathbf{x}^{**}$ after the increase, demanding (at least the same level) all previously demanded items in $\mathbf{x}^*$ for which the price has not risen. This is useful, for example, if the auction is demanding a price path that upholds the gross substitutes property throughout.

Theorem 1 assures us that the gross substitutes property holds when bidders are restricted to the use of bid tables only, as in Stage I of the auction proposed in Day and Raghavan [11], or any isolated bid table auction implementation. To complete the characterization of assignment preferences with respect to the gross substitutes property, we note that the valuation function of Figure 4 maintains the gross substitutes property but cannot be expressed as an assignment preference valuation function. The gross substitutes property follows from the submodularity and positivity of the valuation function. To see that these preferences cannot be expressed as an assignment preference valuation function (or equivalently, as a bid table) notice that the value of any two items is less than the sum of the values for the individual items, thus the values for the individual items must all occur in the same column of a bid table. But now if we attempt to put in a value of 8 into column 2, row a, or a value of 10 into column 2 row b to express $v\{a,b\} = 20$, we either overvalue the bundle $\{a,c\}$ at 21 or the bundle $\{b,c\}$ at 23. There is therefore no way to express these gross substitute preferences as assignment preferences.

FIGURE 5. A bid table auction for which VCG payments are lower than in any Walrasian equilibrium.

	Bidder X			
A	8	0	0	0
B	8	0	0	0
C	0	8	0	0
D	0	8	0	0

	Bidder Y			
A	6	0	0	0
B	0	6	0	0
C	2	0	0	0
D	0	0	0	0

	Bidder Z			
A	0	0	0	0
B	2	0	0	0
C	0	6	0	0
D	6	0	0	0

With Theorem 1 and earlier observations we have the following corollary, where V_{add}, V_{AP}, and V_{sub} denote the classes of valuation functions that have additive preferences, assignment preferences, and gross substitute preferences, respectively.

Corollary 1. $V_{\text{add}} \subset V_{\text{AP}} \subset V_{\text{sub}}$.

Though all containments in Corollary 1 are proper, we note that assignment preferences do retain some of the interesting properties lost between V_{sub} and V_{add}. Notably, we can embed in a bid table auction an example from Gul and Stachetti [15] that demonstrates the VCG outcome (in which truthful bidding is a weakly dominant strategy for all bidders) may have lower payments than the lowest Walrasian Equilibrium. This embedding is shown in Figure 5, providing an example of a bid table auction for which the VCG payments are strictly less than any Walrasian Equilibrium. In one efficient allocation, bidder X gets $\{A, D\}$, bidder Y gets $\{B\}$ and bidder Z gets $\{C\}$, with VCG payments of 12, 2, and 2, respectively. The lowest Walrasian equilibrium price vector is however, $p_A = p_B = p_C = p_D = 6$, charging more to bidders Y and Z than in any VCG outcome. Having established that the VCG outcomes and the lowest Walrasian equilibrium may differ in a bid table auction, we next discuss bid table auctions which achieve each of these outcomes in §§5 and 6, respectively.

Before going on to discuss some nice properties that can be inferred from Theorem 1, however, we first note the extreme limitations on preference expression implied by the theorem's result that $V_{\text{AP}} \subset V_{\text{sub}}$. In particular, the gross substitutes property implies the "no complementarities" condition as described by Gul and Stachetti [14]. This condition forbids any expression of super-additive valuation; for example, a bidder may not require a minimum number of items (other than one) for a bundle to have positive value, or further, if an individual item is added to any bundle it cannot increase the value of the resulting bundle by more than its value if taken individually.

5. VCG Bid Table Auction Implementations

In the sealed-bid Vickrey-Clarke-Groves (VCG) auction (see Clarke [7], Groves [13], Vickrey [25]) each bidder submits her true valuation for every possible bundle of items. A winning allocation is then determined which distributes bundles to bidders so as to maximize total value. To assure that each bidder has the incentive to reveal her total value honestly, she pays not her reported value for the bundle she receives, but this value less an appropriate discount. This VCG discount assures that a bidder does not pay any more than would be necessary to receive this bundle given her opponents honest reports, and is equal to the value of the final allocation minus the maximum value attainable without this bidder. A bidder only decreases her chances of receiving her efficient allocation of items by misreporting, with no possible gain.

The assurance of honest valuation reporting is well known to be the strength of the VCG auction, but as Ausubel and Milgrom [3] point out, there are several drawbacks. Shill bidding occurs when a bidder enters a false identity into the auction in a way that results in an inefficient allocation or alternative set of payments which is preferred by the deceitful bidder. Similarly, Ausubel and Milgrom demonstrate how two or more losing bidders in a VCG auction may increase their bids to become winning without having to pay for this increase. These difficulties with the VCG auction are explored in depth elsewhere, for example by

Rothkopf et al. [24], but we note that these difficulties rely on the existence of complementary items, an impossibility when the gross substitutes condition holds. Theorem 2, stated below, ensures that these problems of collusive behavior and revenue reduction sometimes recognized as drawbacks to the VCG auction have no effect when the gross substitutes property holds.

Theorem 2 (Ausubel and Milgrom [3]). *In an auction where bidders have valuation functions drawn from the set V such that $V_{\text{add}} \subset V$, the following conditions are equivalent:*

(1) $V \subset V_{\text{sub}}$.

(2) *For every profile of bidder valuations drawn from V, adding bidders can never reduce the seller's total revenues in the VCG auction.*

(3) *For every profile of bidder valuations drawn from V, any shill bidding is unprofitable in the VCG auction.*

(4) *For every profile of bidder valuations drawn from V, any joint deviation by losing bidders is unprofitable in the VCG auction.*

Since the hypotheses for Theorem 2 and condition (1) of Theorem 2 are satisfied according to Corollary 1, we may conclude that the bid table auction scenario is one for which the VCG auction attains its full strength. The advantages of the bid table auction scenario over the general VCG auction context are that each bidder may simply submit a bid table to express her preferences (a method that is far more compact than issuing a price for every one of the $2^N - 1$ possible nonempty bundles) and the winner-determination problem can be solved polynomially using LP techniques. Both of these features are important for an auction of many items. Again, we are motivated by the proposed auctions for airport landing slot rights, and notice that at LaGuardia airport, for example, over 800 slots may be available in a single day; an auction for these slots which enumerates all possible bundles would be impossible.

In order to run a *VCG bid table auction*, one must compute both an efficient allocation and the set of VCG payments for all bidders. To determine an efficient allocation, the auctioneer, having received bid tables containing the submitted bids of b_{ijk}, needs to solve the LP:

$$
\begin{aligned}
z_J = \max \quad & \sum_{(i,j,k)\in I\times J\times K_j} b_{ijk} y_{ijk} && \text{(P)}\\
\text{subject to} \quad & \sum_{i\in I} y_{ijk} \le 1, \quad \forall\,(j,k) \text{ with } j\in J \text{ and } k\in K_j,\\
& \sum_{j\in J}\sum_{k\in K_j} y_{ijk} \le \sup_i, \quad \forall\, i\in I,\\
& y_{ijk} \ge 0, \quad \forall\, i,j,k.
\end{aligned}
$$

To determine VCG payments, the auctioneer can solve this problem again without bidder j to find the appropriate discount for bidder j, $z_J - z_{J\setminus j}$, where $z_{J\setminus j}$ denotes the value of the objective value of P with bidder j removed. The assignment corresponding to z_J determines a bundle $\mathbf{x}^j$ with bid value $b_j(\mathbf{x}^j)$ for each bidder. Each bidder in the VCG auction receives $\mathbf{x}^j$ and pays $b_j(\mathbf{x}^j) - (z_J - z_{J\setminus j})$. For the entire auction at most $M+1$ LPs must be solved. Since there are $O(MN^2)$ variables and $O(MN)$ constraints in each LP, and because it is well known that LPs can be solved in polynomial time (as a function of the number of constraints and variables), we conclude that the VCG bid table auction can be solved in polynomial time.

6. Dynamic Bid Table Auctions

Though Theorem 1 allows the application of Theorem 2, implying that because of the gross substitutes property the bid table auction is a good preference format for a VCG

implementation, it also limits the types of possible applications of the bid table auction to those in which there is no complementarity among items. It is possible however to use bid tables in the preliminary (revelation) stages of a hybrid auction concluding with a sealed-bid auction. Theorem 2 and the accompanying discussion by Ausubel and Milgrom [3] make a clear case against the use of the VCG mechanism when complementarities are possible.

Ignoring for now the issue of complements (i.e., maintaining the assumption that bidders have assignment preferences), it seems that the primary drawbacks to the VCG bid table auction are the lack of price discovery (i.e., the transmission of signals to inform bidders of market competition, perhaps affecting their valuations) and privacy preservation (i.e., bidders may not want their preferences to be publicly known). Though privacy preservation may in principle be legally enforced, lack of price discovery may be an inherent concern and discourages the use of a sealed-bid VCG bid table auction. A bidder may not know how to fill out a bid table with honest valuations for agent/item pairs without knowing her opponents' values for various items, and would prefer a dynamic auction to learn about her competition. (This is analogous to the well-known disparity between the English auction and the sealed-bid second-price auction for a single item, in which bidders may prefer the dynamic English auction which allows for price discovery.)

Towards the goal of implementing a dynamic auction for the landing-slot application, we examine the LP dual to P:

$$\min \sum_{i\in I} \sup_i p_i + \sum_{j\in J}\sum_{k\in K_j} s_{jk} \tag{D}$$

$$\text{subject to } p_i + s_{jk} \geq b_{ijk}, \quad \forall\, i,j,k, \tag{7}$$

$$p_i \geq 0, \quad \forall\, i \in I,$$

$$s_{jk} \geq 0, \quad \forall\, j \in J,\, k \in K_j. \tag{8}$$

Problem P has integer optimal solutions (by total unimodularity), and an optimal solution to problem D will have the same objective value. This dual formulation suggests a set of "supporting prices" p_i for each item. If we stipulate that a bidder receiving item i in an optimal solution of P pays p_i, the value of s_{jk} becomes the surplus perceived by bidder j's agent k (abbreviated (j,k)).

The complementary slackness conditions for the primal dual pair P–D are:

$$\forall\, i,j,k \quad y_{ijk} > 0 \;\Rightarrow\; p_i + s_{jk} = b_{ijk} \quad \text{and} \quad p_i + s_{jk} > b_{ijk} \;\Rightarrow\; y_{ijk} = 0; \tag{9}$$

$$\forall\, j,k \quad \sum_{i\in I} y_{ijk} < 1 \;\Rightarrow\; s_{jk} = 0 \quad \text{and} \quad s_{jk} > 0 \;\Rightarrow\; \exists i \text{ such that } y_{ijk} = 1; \tag{10}$$

$$\forall\, i \in I \quad \sum_{j\in J}\sum_{k\in K_j} y_{ijk} < \sup_i \;\Rightarrow\; p_i = 0 \quad \text{and} \quad p_i > 0 \;\Rightarrow\; \sum_{j\in J}\sum_{k\in K_j} y_{ijk} = \sup_i. \tag{11}$$

Each of these conditions (presented in equivalent pairs) carries an economic interpretation reinforcing the validity of the model. If agent (j,k) is awarded item i, condition (9) implies $p_i + s_{jk} = b_{ijk}$, validating our reference to s_{jk} as surplus. If an agent is not awarded an item in an optimal solution, by condition (10) we have $s_{jk} = 0$, and then the constraints (7) become $p_i \geq b_{ijk}$, $\forall\, i \in I$; if an agent is empty-handed at optimality, the optimal dual prices make any item too expensive for this agent to buy. Similarly, the second statement of (10) says that if agent (j,k) perceives any surplus, then it must be the case that (j,k) received an item. Further, if we evaluate the potential surplus that item i would bring to agent (j,k) who receives a different item at optimality, we find $s_{jk} \geq b_{ijk} - p_i$; the price of i is great enough that a change from the item awarded at optimality to i for agent (j,k) does not increase surplus. The last pair of conditions, (11), state simply that an item will have a nonzero price only if all copies of the item are awarded at optimality.

Theorem 3. *Assuming truthful demand reporting and assignment preferences, the allocation* $(\mathbf{x}^1, \mathbf{x}^2, \ldots \mathbf{x}^j \ldots \mathbf{x}^M)$ *given by an optimal solution to P together with prices for items* $\mathbf{p} = (p_1, p_2, \ldots p_i \ldots p_N)$ *given by the corresponding dual solution to D constitutes a Walrasian equilibrium.*

Proof. Suppose not. There is some bidder j and some bundle $\bar{\mathbf{x}}^j$ with j strictly preferring $\bar{\mathbf{x}}^j$ to $\mathbf{x}^j$. Equivalently,

$$b_j(\bar{\mathbf{x}}^j) - \sum_{i \in I} \bar{x}_i^j p_i > b_j(\mathbf{x}^j) - \sum_{i \in I} x_i^j p_i,$$

where $b_j(\mathbf{x}^j)$ and $b_j(\bar{\mathbf{x}}^j)$ are supported by agent sets K and $\overline{K}$, respectively. Complementary slackness conditions (9) provide $b_{ijk} - p_i = s_{jk}$ for each item i assigned to j in $\mathbf{x}^j$, yielding $b_j(\mathbf{x}^j) - \sum_{i \in I} x_i^j p_i = \sum_{k \in K} s_{jk}$. Similarly, constraints (7) yield $\sum_{k \in \overline{K}} s_{jk} \geq b_j(\bar{\mathbf{x}}^j) - \sum_{i \in I} \bar{x}_i^j p_i$. Together this implies

$$\sum_{k \in \overline{K}} s_{jk} > \sum_{k \in K} s_{jk},$$

but with $s_{jk} = 0$ in the optimal solution of D for any agent $k \notin K$, and $s_{jk} \geq 0$ for all agents, we have

$$\sum_{k \in \overline{K} \cap K} s_{jk} > \sum_{k \in K} s_{jk},$$

a contradiction. □

The existence of a Walrasian equilibrium under the gross substitutes property is established by Kelso and Crawford [17]. Among all Walrasian equilibria there exists one that is bidder-optimal, the lowest Walrasian equilibrium. The uniqueness of this bidder-optimal Walrasian equilibrium when the gross substitutes property holds follows naturally from the work of Gul and Stachetti [14], who demonstrate the lattice structure of Walrasian equilibria under gross substitutes. In our case, this existence and uniqueness is guaranteed because the gross substitutes property holds by Theorem 1. This vector of lowest Walrasian equilibrium prices constitute what one might refer to as "good" linear prices, and their existence in the case of bid tables verifies our claim that this is a case for which linear prices make sense. It is known from the work of Demange et al. [12] that versions of the "Hungarian algorithm" (a primal/dual method for solving assignment problems) yield this lowest Walrasian equilibrium in the special case that $A_j = 1, \forall j$, and $\sup_i = 1, \forall i$. We now discuss the generalization to the case of arbitrary integer values for A_j and $\sup_i$.

As in Demange et al. [12], the Hungarian algorithm finds an efficient allocation in a bid table auction and the prices used in its solution provide an optimal solution to the dual problem D, together forming a Walrasian equilibrium, from Theorem 3. (To apply the Hungarian Algorithm directly, it is necessary to give unique names to each identical copy of an item). In general, however, this method produces Walrasian prices that are greater than or equal to the actual lowest Walrasian equilibrium prices for a bid table auction. This is because according to formulation D alone, the price a bidder pays may be determined by one of her own agents, as if her own agents must price-compete among themselves. For example, if the formulation D were used to find prices for the bid table auction represented in Figure 6 (where there is a supply of one for each of two items), the first item would be priced at 3 and the second at 1. The lowest Walrasian Equilibrium prices for this example

FIGURE 6. Self-competition among one's own agents.

	Bidder X			Bidder Y	
A	3	3	A	2	0
B	1	1	B	2	0

are $(1,1)$, however; competition between Bidder X's two agents have drawn up the prices unnecessarily.

To avoid this self-competition problem when finding the lowest Walrasian equilibrium, we introduce the following Dual Pricing Problem DPP. This formulation specifies an LP-characterization of the minimal Walrasian equilibrium prices for a bid table auction. Specifically, given a solution to the primal problem P above with objective value z and allocation determined by optimal y_{ijk} values, we may fix the dual objective from D at its optimal value in constraints (12) and maximize total surplus over all bidders.

$$\max \sum_{j\in J}\sum_{k\in K_j} s_{jk} \tag{DPP}$$

$$\text{subject to } z = \sum_{i\in I} \sup_i p_i + \sum_{j\in J}\sum_{k\in K_j} s_{jk}, \tag{12}$$

$$p_i + s_{jk} = b_{ijk}, \quad \forall\, i,j,k \text{ with } i \to j,k, \tag{13}$$

$$s_{jk} = 0, \quad \forall\, j,k \text{ with } \varnothing \to j,k, \tag{14}$$

$$p_i + s_{jk} \geq b_{ijk}, \quad \forall\, i,j,k \text{ with } \sum_{k\in K_j} y_{ijk} < \sup_i, \tag{15}$$

where we expand our use of the $\to$ notation to express assignment under a specifically chosen efficient solution to problem P; for example, $i \to j,k$ expresses that item i is awarded to bidder j's kth agent in the selected efficient allocation.

We note that the most important distinction between DPP and D comes in constraint set (15) in which only constraints that do not involve self-competition are enforced. That is, form the set of constraints (15) from constraint set (7) of formulation D by removing all constraints $p_i + s_{jk} \geq b_{ijk}$ involving an item i and a bidder j winning all copies of i. This approach is equivalent to resolving the primal allocation problem P and the dual problem D with a Hungarian primal/dual method after lowering all nonwinning entries in a winning row of a bid table to zero when bidder j wins all copies of i. We use the formulation DPP in the following proof as it provides an interesting interpretation as a price adjustment procedure; starting with an efficient allocation and the optimal solution of D, lower the prices on winning bidders as long as no one complains, where any violated constraint from (15) is interpreted as a complaint from another bidder. Knowing that prices may be easily and transparently adjusted to a unique minimal Walrasian equilibrium is a primary benefit of Theorem 1 (gross substitutes). This unique price vector may then be used in a multiround setting, using bid table auctions as a mechanism of demand revelation with distinct meaningful price signals at each round of submission. We now formally prove that we achieve these desirable price signals.

Theorem 4. *Given an optimal solution to the primal problem P with objective z, solving DPP to maximize bidder surplus yields the lowest Walrasian equilibrium price vector $\mathbf{p}^*$.*

Proof. Begin with price vector $\mathbf{p}^1$ which is an optimal solution to D and therefore a Walrasian equilibrium price by Theorem 3. Note that this optimal solution to D gives a feasible solution to DPP with constraints (13) and (14) holding by the strong duality of P and D. Lowering any component p_i^1 and raising all corresponding components s_{jk}^1 by the same amount (where i is assigned to j,k in the solution to P) will have no effect on constraints (12), (13), or (14).

Claim 2. If such a shift from price to surplus does not violate constraints (15) the resulting price vector continues to support a Walrasian equilibrium. If not, some bidder j not assigned all copies of item i (i.e., with $\sum_{k\in K_j} y_{ijk} < \sup_i$) will have an agent (j,k) not assigned i that would prefer item i to whatever item the agent has currently been assigned (if any). This implies that $b_{ijk} - p_i > s_{jk}$ which would violate the corresponding constraint from (15).

We therefore proceed to shift price to surplus until any possible shift causes a violation of some constraint from (15), achieving a price vector $\mathbf{p}^2$.

Claim 3. $\mathbf{p}^2$ is the lowest Walrasian equilibrium price vector. Suppose not: let $\mathbf{p}^3 \neq \mathbf{p}^2$ be the lowest Walrasian price vector. From the lattice theory of Walrasian equilibria when the gross substitutes condition holds (as established in Gul and Stachetti [14]) this Walrasian Equilibrium price is unique, and for every component $p_i^3 \leq p_i^2$. Since $\mathbf{p}^3 \neq \mathbf{p}^2$ there must be some i for which $p_i^3 < p_i^2$. Since our price-to-surplus shifting procedure has terminated, a shift from p_i^2 to p_i^3 must violate some constraint from (15), thus we have the following inequality holding for some j, k with at least one copy of i not assigned to bidder j:

$$p_i^3 + s_{jk} < b_{ijk}.$$

But now at price vector $\mathbf{p}^3$, bidder j who is allocated bundle $\mathbf{x}^j$ prefers the alternative bundle $\mathbf{x}^j + \mathbf{e}^i$, implying that $\mathbf{p}^3$ does not support a Walrasian equilibrium, a contradiction. □

Now that we have shown how the lowest Walrasian price vector will be computed at each round, we propose the *dynamic bid table auction* proceeding in multiple rounds as follows. Accept bid tables from all bidders and determine a winning allocation (solution to P) using any technique. Adjust the prices as suggested in Theorem 4 to find an optimal solution to DPP, and therefore the set of unique lowest Walrasian equilibrium prices. Announce winning prices and allow bidders to adjust their bid tables subject to the following rules:

- Any bid in a nonwinning row of a bid table must be raised at least to the current price for that item (row) plus one price increment, or else it may not be altered for the remainder of the auction.
- A bidder who does not wish to increase an entry to the required amount may increase it to a price below the current price plus one increment. This is the bidder's "last-and-best" offer.

The auction then continues by computing a new set of winning bids and prices, and the process repeats until no one wishes to raise any bid table entries any further.

To show that this procedure progresses to a desirable equilibrium, assume that each bidder perceives a set of maximum bid table values, v_{ijk}. We would expect to see these entries from honest bidders in the direct revelation VCG as in §5, and now show that the dynamic game converges to the same outcome as the direct revelation game, given an assumption of straightforward bidding. Though this behavioral assumption is strong, there is evidence in practice and from the relevant literature (especially from Ausubel et al. [4]) that an appropriate bidding *activity rule* will encourage straightforward bidding behavior. (Ausubel et al. [4] provide a concrete example of a practical activity rule. In general, an activity rule reduces the eligibility of a bidder to bid in future rounds if she does not bid aggressively enough in the present.)

For the dynamic bid table auction scenario, we will say that a bidder bids straightforwardly if she increases a bid table b_{ijk} in a nonwinning row by the minimal increment whenever the potential surplus $v_{ijk} - p_i$ for agent (j, k) is greater than the actual current surplus for agent (j, k), $\bar{s}_{jk} = v_{\bar{\imath}jk} - p_{\bar{\imath}}$, where $\bar{\imath}$ is the item currently awarded to agent (j, k). Here the modifier *actual* and "bar" over the s emphasize that this is the actual surplus as perceived by the bidder, not the apparent surplus which may be computed using her revealed information: $s_{jk} = b_{\bar{\imath}jk} - p_{\bar{\imath}}$. In either case, this surplus will be zero if no item is awarded to agent k. We also assume that if a straightforward bidder is asked to raise a bid table value above v_{ijk}, then she will raise it to v_{ijk}, reporting her true valuation as she loses eligibility to alter the entry further.

Theorem 5. *A dynamic bid table auction with straightforward bidders* (A) *terminates at an efficient equilibrium of the direct revelation bid table auction, and* (B) *achieves the unique lowest Walrasian equilibrium prices.*

Proof of Part A. First, we show that the allocation at termination is an efficient outcome of the direct revelation game. Suppose not. Let A be the allocation at termination of the dynamic auction with current bid table values b_{ijk}. By supposition there is an allocation $\bar{A}$ such that

$$\sum_{\bar{A}} y_{ijk} v_{ijk} > \sum_{A} y_{ijk} v_{ijk} \tag{16}$$

where the summation over an allocation signifies summation over all i, j, k with values of y_{ijk} given by that allocation. Because we have assumed that bidders bid straightforwardly and that the dynamic auction has terminated, for any bid table column j, k which is allocated item i under allocation A and $\bar{\imath}$ under $\bar{A}$, it must be the case that $v_{\bar{\imath}jk} - p_{\bar{\imath}} \leq v_{ijk} - p_i$ (if not the straightforward bidder would want to continue bidding on $\bar{\imath}$). This inequality also holds (reflexively) for any agent j, k that is awarded the same item under both allocation A and $\bar{A}$. For any j, k allocated item $\bar{\imath}$ under $\bar{A}$ which is not allocated an item under A, this condition becomes $v_{\bar{\imath}jk} - p_{\bar{\imath}} \leq 0$. Finally, by individual rationality we also have that $0 \leq v_{ijk} - p_i$ for any i allocated to j, k under A (particularly we take this inequality for any columns j, k which are for agents receiving items in both allocations, allocated items under A but not under $\bar{A}$). We then sum these three sets of inequalities, selecting (and multiplying by y_{ijk}) the appropriate one for each agent j, k who receives items in either allocation A or $\bar{A}$, or both, yielding:

$$\sum_{\bar{A}} y_{ijk}(v_{ijk} - p_i) \leq \sum_{A} y_{ijk}(v_{ijk} - p_i)$$

$$\sum_{\bar{A}} y_{ijk} v_{ijk} \leq \sum_{A} y_{ijk} v_{ijk}, \tag{17}$$

with the second inequality following since each item is allocated exactly once in each allocation, allowing us to cancel the sum of all p_is from each side. But (17) contradicts (16), with the desired result following.

Proof of Part B. Next we show that the prices at termination of the dynamic schedule auction are the lowest Walrasian equilibrium prices for the direct revelation schedule auction. Using Theorem 4, this is equivalent to showing that the solution to the pricing problem DPP using the value $\bar{z}$ (computed using values from v_{ijk}) has equal values for all p_i to the solution of DPP using value z (computed using b_{ijk}).

Given that the optimal allocation is unchanged by increasing all b_{ijk} values to their reservation point v_{ijk} from Part A, increase all b_{ijk} values to v_{ijk} and simultaneously increase every surplus s_{jk} by the same amount wherever i is allocated to j, k. We claim that this provides a solution to DPP using value $\bar{z}$ with identical values of all p_i, as desired. Clearly, the simultaneous shift of s_{jk} values with b_{ijk} values upholds DPP constraints (12) and (13). Constraints (14) are upheld since s_{jk} only changes for columns which win items, while a violation of a constraint from (15) would imply a violation of a termination condition. Since none of the constraints of DPP are violated, our new solution must be feasible to DPP. Since any increase of the DPP objective function must be accompanied by an equivalent increase to the primal objective z, and because we have achieved all such increase as surplus, we may be assured that our new objective function is optimal. Since no p_i values have changed we have the same lowest Walrasian equilibrium price vector at the termination of the dynamic bid table auction and in the direct revelation bid table auction, as desired.

Part B of this proof demonstrates the desirable privacy-preservation property of the dynamic bid table auction relative to the static revelation version: the actual surplus each bidder perceives need not be revealed in the dynamic case, only that surplus is at least one increment. By making the bid increment smaller we may in this way approach maximal privacy-preservation. □

The results of this section hold under the assumption that each bidder's preferences are modeled accurately in bid tables, an assumption which we have shown to be more restrictive than the gross substitutes property. Recent trends in auction design suggest that the use of auctions which work well under the gross substitutes property (e.g., those of Gul and Stachetti [15], Ausubel [2]) may be used to reveal information necessary for bidders to make decisions in an auction allowing for more general expression of preferences, those for which gross substitutes does not hold. At the end of a dynamic bid table auction, each bidder should be comfortable that he has bid enough on individual items without being exposed to the risk of paying too much for substitute items. Then, the auction may proceed to a subsequent phase in which a bidder submits combinatorial bids on bundles for which her valuation is greater than the sum of the individual item prices. For a further discussion of a multistage combinatorial auction with the dynamic bid table auction as a first step see Day and Raghavan [11].

To understand this idea in a bit more detail within the current context, imagine a bidder participating in a dynamic bid table auction that will be followed by a more general expression of preferences in the later stage(s) of a hybrid auction. This bidder may select b_{ijk} values for her bid table such that any feasible solution to (AP) for a particular bundle $\mathbf{x}$ will be less than or equal to her desired bid on the bundle $b_j(\mathbf{x})$. Then the solution to (AP), which we can call $b_j^{\mathrm{AP}}(\mathbf{x})$, will be a lower bound on $b_j(\mathbf{x})$ for any $\mathbf{x}$. When all bidders use this technique of bid table expression to communicate *lower* bounds on bundle values, the resulting dual constraints of the form given by (15) can be added appropriately to get $\sum_{i\in I} x_i p_i + \sum s_{jk} \geq b_j^{\mathrm{AP}}(\mathbf{x})$, $\forall\, j \in J$, $\forall\, \mathbf{x} \in \mathcal{S}$. Comparing to constraints of the form (4), we note that $s_j - \sum s_{jk} = b_j(\mathbf{x}) - b_j^{\mathrm{AP}}(\mathbf{x})$ and thus the dynamic bid table auction results in equilibrium prices of the form described by problem (WDP–D), but based on lower-bound bids on each bundle rather than actual (eventual) bids, with the difference between a bundle's actual value and its lower bound hidden as additional, unobservable surplus. The goal of the next phase of any hybrid auction would be to extract additional information about this yet unobserved surplus to increase auction efficiency. The tighter the lower bounds provided by the selection of b_{ijk} values, the better the approximation of the actual efficient solution obtained by this approximation in a polynomially computable phase.

Now, comparing the ability of the SAA to perform the same goal in a hybrid auction, we see that the final prices in a SAA can be modeled as those of a bid table auction with the restriction than only bid table entries on the diagonal may be positive. Thus with a smaller space of possible b_{ijk} entries, the lower bounds on bundle values provided by SAA must be inferior to those provided by a general dynamic bid table auction. Further, actual SAA implementations allow the current winner to set the current price on an item in intermediate rounds, allowing for collusive signaling as observed by Cramton and Schwartz [8]. But since the current winning bidder of an item in a dynamic bid table auction does not set the price observed by competitors, this form of collusive behavior is impossible in a dynamic bid table auction.

7. Conclusions

This paper focuses primarily on a special case of a combinatorial auction for which the size of communication grows polynomially in the number of items being auctioned, and for which the winner-determination problem can be solved in time polynomial in the size of this communicated input. Implicitly, we ask: how can a simple compact representation of preferences (price-vectors) be combined to form more elaborate statements of preference? This question follows somewhat naturally from the economics literature of Kelso and Crawford [17], who introduce unit-demand bidders, each with preferences described by a price-vector, in a model of the job-market, in which an employee can accept at most one job. Kelso and Crawford also introduce the gross substitutes property, which has become fundamental in

the study of auctions. The strength of this concept in categorizing preferences and guaranteeing the existence of a unique Walrasian linear-price equilibrium has influenced several authors, including Ausubel and Milgrom [3] and Gul and Stachetti [15], all of whom provide foundational work for the research presented here.

Other results on unit-demand bidders are given by Demange et al. [12], leading naturally into our own investigation of bid tables. Indeed, the case of unit-demand bidders is well studied, and we note that the current paper generalizes the Walrasian equilibrium results of Demange et al. [12] (as presented, for example, by Roth and Sotomayor [23]), from the case of unit-demand bidders to the case in which each bidder is represented by multiple unit-demand agents.

Representing bidders by multiple unit-demand agents results in the fairly natural and easy-to-read bid table format, causing us to wonder, why hasn't this been investigated before? We note that among the incentive properties of their unit-demand bidder auction model, Roth and Sotomayor [23] show that a bidder cannot benefit from shill bidding (having someone else join the auction to distort its outcome). This may seem to imply that representing a bidder by multiple unit-demand agents would be unrewarding, but the model under which this property was proven maintains the assumption of a unit-demand valuation function for each bidder. We, on the other hand, find use for the bid table format (i.e., assignment preferences) within the more general context of multiunit demand.

How general is the preference expression afforded by the bid table format in the multiunit demand context? We have provided a powerful characterization: assignment preferences (those which can be written in bid tables) are properly contained in the class of gross substitutes valuation functions. Applying a result of Gul and Stachetti [14] based on this characterization, the gross substitutes property elucidates the greatest strength of a bid table auction: unique lowest Walrasian equilibrium price signals can be computed at each round of submission. The computation of these attractive linear price signals is facilitated by our constrained optimization approach, which allows us to recognize and neutralize "self-competition" constraints. This new approach overcomes the failure of the Hungarian algorithm to provide the truly lowest Walrasian equilibrium prices in this context, as it does in the unit-demand bidder context.

While the gross substitutes property does indeed provide several strengths of the bid table environment, it also clearly exposes its weaknesses. Most notably, preferences for complementary bundles cannot be expressed in bid tables alone. We therefore demonstrated the efficacy of a dynamic bid table auction which can be used as the first stage of a hybrid auction procedure, capturing substitutable preferences in bid tables and deferring to a later package auction phase for complementary expression. Further details on the design of such a hybrid auction are given in Day and Raghavan [11], but with the current work we have established much of the foundational theory for that design, allowing us to implement a better revelation phase for a combinatorial hybrid auction. These results have included algorithms for computing better linear price signals, and for upholding the gross substitutes property in a strict sense, where earlier and more general algorithms fail to do so.

References

[1] G. Anandalingam, R. Day, and S. Raghavan. The landscape of electronic market design. *Management Science* 51(3):316–327, 2005.

[2] L. Ausubel. An efficient dynamic auction for heterogeneous commodities. *American Economic Review* 96(3):602–629, 2006.

[3] L. Ausubel and P. Milgrom. Ascending auctions with package bidding. *Frontiers of Theoretical Economics* 1(1):1–42, 2002.

[4] L. Ausubel, P. Cramton, and P. Milgrom. The clock-proxy auction: A practical combinatorial auction design. P. Cramton, Y. Shoham, and R. Steinberg, eds. *Combinatorial Auctions*, Chapter 5. MIT Press, Cambridge, MA, 115–138, 2006.

[5] M. Ball, G. Donohue, and K. Hoffman. Auctions for the safe efficient and equitable allocation of national airspace systems resources. P. Cramton, Y. Shoham, and R. Steinberg, eds. *Combinatorial Auctions*, Chapter 22. MIT Press, Cambridge, MA, 507–538, 2006.

[6] S. Bikhchandani and J. Ostroy. From the assignment model to combinatorial auctions. P. Cramton, Y. Shoham, and R. Steinberg, eds. *Combinatorial Auctions*, Chapter 8. MIT Press, Cambridge, MA, 189–214, 2006.

[7] E. Clarke. Multipart pricing of public goods. *Public Choice* 8:19–33, 1971.

[8] P. Cramton and J. Schwarz. Collusive bidding: Lessons from the FCC spectrum auctions. *Journal of Regulatory Economics* 17:229–252, 2000.

[9] P. Cramton, Y. Shoham, and R. Steinberg. *Combinatorial Auctions*. MIT Press, Cambridge, MA, 2006.

[10] R. Day. Expressing preferences with price-vector agents in combinatorial auctions. Ph.D. thesis, University of Maryland, College Park, 2004.

[11] R. Day and S. Raghavan. A combinatorial procurement auction featuring bundle price revelation without free riding. *Decision Support Systems* 44(3):621–640, 2008.

[12] G. Demange, D. Gale, and M. Sotomayor. Multi-item auctions. *Journal of Political Economy* 94(4):863–872, 1986.

[13] T. Groves. Incentives in teams. *Econometrica* 41:617–631, 1973.

[14] F. Gul and E. Stachetti. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory* 87:95–124, 1999.

[15] F. Gul and E. Stachetti. The English auction with differentiated commodities. *Journal of Economic Theory* 92:66–95, 2000.

[16] J. Hatfield and P. Milgrom. Matching with contracts. *American Economic Review* 95(4):913–935, 2005.

[17] A. Kelso and V. Crawford. Job matching coalition formation and gross substitutes. *Econometrica* 50:1483–1504, 1982.

[18] B. Lehmann, D. Lehmann, and N. Nisan. Combinatorial auctions with decreasing marginal utilities. *Games and Economic Behavior* 55(2):270–296, 2006.

[19] D. Lehmann, R. Müller, and T. Sandholm. The winner determination problem. P. Cramton, Y. Shoham, and R. Steinberg, eds. *Combinatorial Auctions*, Chapter 12. MIT Press, Cambridge, MA, 297–318, 2006.

[20] R. Müller. Tractable cases of the winner determination problem. P. Cramton, Y. Shoham, and R. Steinberg, eds. *Combinatorial Auctions*, Chapter 13. MIT Press, Cambridge, MA, 319–336, 2006.

[21] G. Nemhauser and L. Wolsey. *Integer and Combinatorial Optimization*. John Wiley and Sons, New York, 1999.

[22] N. Nisan. Bidding languages for combinatorial auctions. P. Cramton, Y. Shoham, and R. Steinberg, eds. *Combinatorial Auctions*, Chapter 9. MIT Press, Cambridge, MA, 215–232, 2006.

[23] A. Roth and M. Sotomayor. *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Cambridge University Press, Cambridge, UK, 1992.

[24] M. Rothkopf, T. Teisberg, and E. Kahn. Why are Vickrey auctions rare? *Journal of Political Economy* 98:94–109, 1990.

[25] W. Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance* 16:8–37, 1961.